Sequential Weak Approximation for Maps of Finite Hessian Energy

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Abstract : *Consider the space $W^{2,2}(\Omega;N)$ of second order Sobolev mappings v from a smooth domain $\Omega\subset\mathbb{R}^m$ to a compact Riemannian manifold N whose Hessian energy $\int_\Omega|\nabla^2 v|^2\,dx$ is finite. Here we are interested in relations between the topology of N and the $W^{2,2}$ strong or weak approximability of a $W^{2,2}$ map by a sequence of smooth maps from Ω to N. We treat in detail $W^{2,2}(\mathbb{B}^5,\mathbb{S}^3)$ where we establish the sequential weak $W^{2,2}$ density of $W^{2,2}(\mathbb{B}^5,\mathbb{S}^3)\cap\mathcal{C}^\infty$. The strong $W^{2,2}$ approximability of higher order Sobolev maps has been studied in the recent preprint [BPV] of P. Bousquet, A. Ponce, and J. Van Schaftigen. For an individual map $v\in W^{2,2}(\mathbb{B}^5,\mathbb{S}^3)$, we define a number $L(v)$ which is approximately the total length required to connect the isolated singularities of a strong approximation u of v either to each other or to $\partial\mathbb{B}^5$. Then $L(v)=0$ if and only if v admits $W^{2,2}$ strongly approximable by smooth maps. Our critical result, obtained by constructing specific curves connecting the singularities of u, is the bound $L(u)\leq c\int_{\mathbb{B}^5}|\nabla^2 u|^2\,dx$. This allows us to construct, for the given Sobolev map $v\in W^{2,2}(\mathbb{B}^5,\mathbb{S}^3)$, the desired $W^{2,2}$ weakly approximating sequence of smooth maps. To find suitable connecting curves for u, one uses the twisting of a u pull-back normal framing of a suitable level surface of u.*

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I Introduction

To motivate our specific work on weak sequential approximability of $W^{2,2}$ maps from $\mathbb{B}^5$ to $\mathbb{S}^3$, we will first describe briefly the background and general problems. Let (N, g) be a compact Riemannian manifold. Via the Nash embedding theorem, one may assume that N is a submanifold of some Euclidian space $\mathbb{R}^\ell$ and that the metric g is induced by this inclusion. One then has, for any open subset Ω of $\mathbb{R}^m$, $k \in \mathbb{N}$, and $p > 1$, the nonlinear space of kth order, Sobolev maps

$$W^{k,p}(\Omega, N) \; = \; \{u \in W^{k,p}(\Omega, \mathbb{R}^\ell) \; : \; u(x) \in N \text{ for almost every x} \in \Omega\},$$

where $W^{k,p}(\Omega, \mathbb{R}^\ell)$ denotes the Banach space of $\mathbb{R}^\ell$-valued, order k, Sobolev functions on Ω with norm $\|u\|_{W^{k,p}} \; = \; \left[\sum_{j=0}^k \left(\int_\Omega |\nabla^j u|^p \, dx\right)^{2/p}\right]^{1/2}$.

I.1 Strong Approximation

A basic question concerning the spaces $W^{k,p}(\Omega, N)$ is the *approximability of these maps by a sequence of smooth maps* of Ω into N. The issue involves the possible discontinuities in a Sobolev map because any continuous Sobolev map may be approximated strongly in the Sobolev norm. In fact, here ordinary smoothing [A] gives both uniform and $W^{2,2}$ strong approximation by an $\mathbb{R}^\ell$-valued smooth Sobolev function whose image lies in a small neighborhood of N; then composing this with the nearest-point projection to N gives the desired smooth strong approximation with image in N. It was first observed in [SU] that for $n = 2$, a $W^{1,2}$ map (which may fail to have a continuous representative) admits strong $W^{1,2}$ approximation by smooth $W^{1,2}$ maps into N. However for $n = 3$, [SU] also showed that the specific singular Sobolev map $x/|x| \in W^{1,2}(\mathbb{B}^3, \mathbb{S}^2)$ is *not* strongly approximable in $W^{1,2}$ by smooth maps from $\mathbb{B}^3$ to $\mathbb{S}^2$. For first order Sobolev maps, the general problem of strong $W^{1,p}$ approximability was treated by F. Bethuel in [Be2], which (with [BZ]) shows that:

$$W^{1,p}(\mathbb{B}^m, N) \text{ is the sequential strong } W^{1,p} \text{ closure of } \mathcal{C}^\infty(\mathbb{B}^m, N) \quad \Longleftrightarrow \quad \Pi_{[p]}(N) = 0 \; .$$

Here $[p]$ is the greatest integer less than or equal to p. F. Hang and F. H. Lin, in [HaL1] and [HaL2], updated these results with some new proofs and corrections, which account for the role played by the topology of the domain in approximability questions. See [HaL2], Th.1.3 for the precise conditions on the domain. There are many other interesting works on strong approximability of first order Sobolev maps by smooth maps, e.g. [Be1], [BCL], [Hj], [BCDH], [BBC], [BZ]. Generalization of the strong approximability results of [Be2], [HaL1], and [HaL2] to higher order Sobolev mappings has been treated by P. Bousquet, A. Ponce, and J. Van Schaftigen in [BPV] which (with [BZ]) shows that

$$W^{k,p}(\mathbb{B}^m, N) \text{ is the sequential strong } W^{k,p} \text{ closure of } \mathcal{C}^\infty(\mathbb{B}^m, N) \quad \Longleftrightarrow \quad \Pi_{[kp]}(N) = 0\ .$$

I.2 Sequential Weak Approximation

The space $W^{k,p}(\Omega, N)$ also inherits the *weak topology* from $W^{k,p}(\Omega, \mathbb{R}^\ell)$. A Sobolev map in $W^{k,p}(\Omega, N)$ that is not $W^{k,p}$ strongly approximable by smooth maps may be $W^{k,p}$ *weakly approximable* by a sequence of smooth maps. For example, the map $x/|x| \in W^{1,2}(\mathbb{B}^3, \mathbb{S}^2)$ *is* weakly approximable in $W^{1,2}$ by some sequence u_i of smooth maps. The well-known construction of such a u_i involves changing $x/|x|$ in a thin cylindrical tunnel U of width $1/i$ extending from the origin $(0, 0, 0)$ to a point on $\partial\mathbb{B}^3$. To prove the weak $W^{1,2}$ weak convergence of u_i to $x/|x|$, the key point of the construction is to keep the energies $\int_{\mathbb{B}^3} |\nabla u_i|^2 dx$ bounded independent of i.

To find an example of a map $v \in W^{1,p}(\mathbb{B}^m, N)$ which does not have a *weakly* approximating sequence of smooth maps, we need both $p < m$ and $\Pi_{[p]}(N) \neq 0$. Then, *if p is not an integer*, we simply choose, as in [Be2], any map v which fails to have *strong* smooth approximations. Assuming for contradiction that this map v did admit some weak approximation by smooth maps v_i, then, for every point $a \in \mathbb{B}^m$, Fubini's theorem and Sobolev embedding (because $p > [p]$), would give *strong* convergence of the restrictions $v_i|\mathbb{S}_a$ to $v|\mathbb{S}_a$ for almost every $[p]$ dimensional Euclidean sphere $\mathbb{S}_a$ centered at a in $\mathbb{B}^m$. Then by the smoothness of v_i and by [W], the corresponding homotopy classes $[\![v|\mathbb{S}_a]\!]$ would all vanish. But Bethuel showed in [Be1] that precisely this local vanishing homotopy condition on $[p]$ spheres would imply that v *does* admit strong smooth approximation, a contradiction.

For integer p the following question is still open:

For any any compact manifold N, any integers k, $m \geq 1$ and any integer $p \geq 2$, is every Sobolev map $v \in W^{k,p}(\mathbb{B}^m, N)$ actually $W^{k,p}$ weakly approximable by a sequence of smooth maps?

This sequential weak density of smooth maps has been verified in the following cases:

(1) [BBC], [ABL] : $W^{1,p}(\mathbb{B}^m, \mathbb{S}^p)$.

(2) [Hj]: $W^{1,p}(\mathbb{B}^m, N)$ with N being simply $p - 1$ connected (i.e. $\Pi_j(N) = 0$ for $0 \leq j \leq p - 1$).

(3) [Pa]: $W^{1,1}(M, N)$ with M and N being arbitrary smooth manifolds with $\partial N = \emptyset$ (weak convergence has to be understood in a *biting* sense here)

(4) [PR]: $W^{1,2}(\mathbb{B}^m, N)$. (See also [Ha] concerning the role of the topology of M in $W^{1,2}(M, N)$.)

See also a presentation of these results in [Ri]. Another case is the main result of the present paper:

Theorem V. *Any map in $W^{2,2}(\mathbb{B}^5, \mathbb{S}^3)$ may be approximated in the $W^{2,2}$ weak topology by a sequence of smooth maps.*

In §I.4 below, we will explain how we came to study maps from $\mathbb{B}^5$ to $\mathbb{S}^3$ and to look for $W^{2,2}$ estimates. But first we review a few of the ideas that were developed to study sequential weak convergence of smooth maps. The space $W^{1,2}(\mathbb{B}^3, \mathbb{S}^2)$ was studied extensively in the late eighties and early nineties with many works, e.g. [HL], [BCL], [BZ], [BBC], [GMS1]. The concrete results of these many works has led to some analogous results and many conjectures for more general k, n, p, and N. To sequentially weakly approximate a map $v \in W^{1,2}(\mathbb{B}^3, \mathbb{S}^2)$, one first finds a $W^{1,2}$ *strong* approximation from the family $\mathcal{R}_0(\mathbb{B}^3, \mathbb{S}^2)$ of maps $u \in W^{1,2}(\mathbb{B}^3, \mathbb{S}^2)$ which are smooth away from some finite set $\operatorname{Sing} u$. In particular, we may assume $\int_{\mathbb{B}^3} |\nabla u|^2 dx \leq 2 \int_{\mathbb{B}^3} |\nabla v|^2 dx$. Here the topology of u near a point $a \in \operatorname{Sing} v$ is given by the integer $d(a) = \operatorname{degree}[u|\partial\mathbb{B}_\varepsilon(a)]$, which is independent of a.e. small ε. Then to get the desired completely smooth weak approximate, it is necessary to essential cancel the singularities of u. One does this by finding a one-chain or "connection" Γ_u with $\partial\Gamma_u$ in $\mathbb{B}^3$ being $\sum_{a \in \operatorname{Sing} v} d(a)[\![a]\!]$ and with the rest of $\partial\gamma_u$

lying in ∂B^3. Then, as with the argument for $x/|x|$, one constructs smooth maps u_i by making changes in tunnels of radius $1/i$ centered along the connection. To keep the $|\nabla u_i|^2$ integrals bounded, one needs to find a bound for the total length of the connection Γ_u that depends only on v, and is independent of the approximating u. Here one may find a suitable connection by using the coarea formula. This gives a good level curve of u which connects the singularities to each other and to $\partial\mathbb{B}^3$ and which has length bounded by $\int_{\mathbb{B}^3} |\nabla u|^2 dx$, which has the independent bound $2\int_{\mathbb{B}^3} |\nabla v|^2 dx$.

The first part of this argument, the strong $W^{1,2}$ approximation of an arbitrary Sobolev map $v \in W^{1,2}(\mathbb{B}^3, \mathbb{S}2)$ by a map $u \in \mathcal{R}_0(\mathbb{B}^3, \mathbb{S}^2)$ has been generalized in [Be2] to all $W^{1,p}(\mathbb{B}^m, N)$ and recently in [BPV] to all $W^{k,p}(\mathbb{B}^m, N)$. Here one gets strong approximation by maps in $\mathcal{R}_{m-[p]-1}(\mathbb{B}^m, N)$ (respectively, $\mathcal{R}_{m-[kp]-1}(\mathbb{B}^m, N)$ which are smooth with singularities lying in finitely many affine planes of dimension $m - [p] - 1$ (respectively, $m - [kp] - 1$).

However, the second part involving canceling the singularities of u has proven very challenging for generalization. One roughly needs an $m - [p]$ (respectively, $m - [kp] - 1$) dimensional connection which has mass bounded in terms of the energy of u and which the connects the singularity. Even with the connection, one still has to construct the bounded energy, smooth approximate.

I.3 Topological Singularity and Bubbling

In this supercritical dimension $m > kp$, we see that studying sequential $W^{k,p}$ weak smooth approximation in $W^{k,p}(\mathbb{B}^m, N)$ leads to questions about the relationship between the possible energy drop, $\int_{\mathbb{B}^m} |\nabla^k u|^p dx < \liminf_{i\to\infty} \int_{\mathbb{B}^m} |\nabla^k u_i|^p dx$, of a $W^{k,p}$ weakly convergent sequence $u_i \in W^{k,p}(\mathbb{B}^m, N) \cap \mathcal{C}^\infty$ and the possible singularities of its weakly convergent limit $u \in W^{k,p}(\mathbb{B}^m, N)$.

For $0 \neq \alpha \in \Pi_{kp}(N)$, we say a point $a \in \mathbb{B}^m$ is a type α *topological singularity* of a $W^{k,p}$ map u if there is an $kp+1$ dimensional affine plane P containing a so the restrictions of u to a.e. small kp sphere $P \cap \partial\mathbb{B}_\varepsilon(a)$ induce (i.e. in the sense of [W]) the homotopy class α. Following the $W^{k,p}$ strong density of the partially smooth maps $\mathcal{R}_{m-kp-1}(\mathbb{B}^m, N)$ in $W^{k,p}(\mathbb{B}^m, N)$, one expects the topological singularities, with their types as coefficients, to form a chain S_u having dimension $m - kp - 1$ and having coefficients in the group $\Pi_{kp}(N)$. Recall the criterion of [Be1] that the vanishing of this "u topological singularity" chain(that is the vanishing of such homotopy classes for a.e. such restrictions at every $a \in \mathbb{B}^m$) is equivalent the $W^{k,p}$ strong approximability of u by smooth maps.

Also for $0 \neq \alpha \in \Pi_{kp}(N)$, the restrictions of u_i to generic affine kp planes can, as $i \to \infty$ have $|\nabla^k|^p$ energy concentration at an isolated point b with an associated topological change corresponding to a type beta "bubble". Putting such points together with their bubble types as coefficients should give a "u_i bubbled" chain B_{u_i} that has dimension $m-kp$, that has coefficient group $\Pi_{kp}(N)$, and that is carried by the $|\nabla^k(\cdot)|^p$ energy concentration set of the sequence.

Using these vague definitions, one has the vague general conjecture:

Relative to $\partial\mathbb{B}^m$, the boundary of the u_i bubbled chain B_{u_i} equals the u topological singularity chain S_u.

The vagueness here concerns the precise definition of chain and boundary operation, and how one precisely obtains the bubbled chain B_{u_i} from the sequence u_i and the topological singular chain S_u from u. From the cases we know, it is clear there is no single answer; it depends on the Sobolev space $W^{k,p}(\mathbb{B}^m, N)$, in particular the group $\Pi_{kp}(N)$.

In the special case $W^{1,2}(\mathbb{B}^3, \mathbb{S}^2)$, the relevant homotopy group is $\Pi_2(\mathbb{S}^2) \simeq \mathbb{Z}$, and [BBC] and [GMS1] show that this 1 chain is precisely an integer-multiplicity 1 dimensional rectifiable current of finite mass (but possibly infinite boundary mass). The special case was essentially generalized to $k = 1$, $N = \mathbb{S}^p$ in [GMS2] and [ABO]. Here, the bubbled chain, now of dimension $m - p$, is again a rectifiable current.

The paper [HR1] treated $W^{1,3}(\mathbb{B}^3, \mathbb{S}^2)$. The relevant homotopy group is $\Pi_3(\mathbb{S}^2)$, which is again isomorphic to $\mathbb{Z}$. Here the bubbled 1 chain was shown to be possibly of infinite mass, and the notion of a "scan" was invented to describe precisely compactness and boundary properties. The paper [HR2] has able to handle bubbling in weak limits of smooth maps that corresponds to *any* nonzero homotopy class in the infinite nontorsion part of $\Pi_p(N)$. In this situation, the homotopy class of a map w on the sphere $\mathbb{S}^{m-1}$ can again be described using a differential $m-1$ form Φ_w on $\mathbb{S}^{m-1}$. The form is derived by a special algebro-combinatoric construction (depending on the rational homotopy class) involving a family of w pullbacks of forms on N and their "d^{-1} integrals". For example, in case $w : \mathbb{S}^3 \to \mathbb{S}^2$, $\Phi_w = w^\# \omega_{\mathbb{S}^2} \wedge d^* \Delta^{-1} w^\# \omega_{\mathbb{S}^2}$. In general, this representation by a finite family of differential forms allows useful energy estimates involving certain Gauss integrals. For a weakly convergent sequence of smooth maps, the bubbled chain, which cannot usually be represented as a finite mass current, can be understood precisely as a rectifiable scan whose boundary is given by the topological singularities of the limit Sobolev map. Though we have a somewhat satisfactory description of bubbling and topological singularity in all nontorsion cases, the question of sequential weak density in these cases are still not resolved, even for the case $W^{1,3}(\mathbb{B}^4, \mathbb{S}^2)$.

Unfortunately representations of a homotopy class by differential forms are not available for *torsion* classes. In particular, if the relevant homotopy group of N is completely torsion, then one requires other techniques to get energy estimates needed for questions about weak limits of smooth maps. The first such case is $W^{1,2}(\mathbb{B}^3, \mathbb{R}P^2)$, and was treated in [PR] . Here $\Pi_2(\mathbb{RP}^2) \simeq \mathbb{Z}_2$, and, a main result, is that *smooth maps are* $W^{1,2}$ *sequentially weakly dense* because $p = 2$.

I.4 $W^{2,2}(\mathbb{B}^5, \mathbb{S}^3)$

The present paper started with the modest goal of understanding analytic estimates for maps of $w : \mathbb{S}^4 \to \mathbb{S}^3$ so as to understand weak convergence and sequential weak density for smooth maps from $\mathbb{B}^5$ to $\mathbb{S}^3$. Here, the appropriate homotopy group is $\Pi_4(\mathbb{S}^3)$, which is isomorphic to $\mathbb{Z}_2$. But geometric descriptions of this homotopy class of v are not very simple. As discussed in Sections IV.2 and IV.3 below, they involve considering, for a smooth approximation u of v, the total twisting of a u-pullback normal framing upon circulation around a generic fiber $u^{-1}\{y\}$. The twisting of the normal frame leads to an element of $\Pi_1(\mathbb{SO}(3)) \simeq \mathbb{Z}_2$. To analytically compute such a twisting involves integration of a derivative of a pull-back framing, hence a *second* derivative of the original map. So it is natural for this homotopy group to try to look for estimates in terms of the Hessian energy.

A representative of the single nonzero element in $\Pi_4(\mathbb{S}^4, \mathbb{S}^3)$ is the suspension of the Hopf map, $\mathbb{SH} : \mathbb{S}^5 \to \mathbb{S}^3$, described explicitly in the next section. In Section II below, we slightly adapt [BPV], Th.5 by defining the subfamily

$$\mathcal{R} = \{u \in \mathcal{R}_0(\mathbb{B}^5, \mathbb{S}^3) \ : \ u \equiv \mathbb{SH}\left(\frac{x-a}{|x-a|}\right) \text{ on } \mathbb{B}_{\delta_0}(a) \setminus \{a\} \text{ for all } a \in \operatorname{Sing} u \text{ and some } \delta_0 > 0\} \quad \text{(I.1)}$$

and then proving:

Lemma III.2 *The family $\mathcal{R}$ is $W^{2,2}$ strongly dense in $W^{2,2}(\mathbb{B}^5, \mathbb{S}^3)$.*

I.5 Connection Length

Given a finite subset A of $\mathbb{B}^5$, one may define $\mathbb{Z}_2$ *connection for* A (relative to $\partial\mathbb{B}^5$) as a finite disjoint union Γ of finite length arcs embedded in $\overline{\mathbb{B}^5}$ whose union of endpoints is precisely $A \cup (\Gamma \cap \partial\mathbb{B}^5)$. Thus, each point of A is joined by a unique arc in Γ to either another point of A or to a point of $\partial\mathbb{B}^5$.

It will simplify some constructions to use a *minimal* $\mathbb{Z}_2$ *connection* for A, that is, one having least length. It is not difficult to verify the existence and structure of a minimal $\mathbb{Z}_2$ connection for A. It simply consists of the disjoint union of finitely many closed intervals in $\mathbb{B}^5$ and finitely many radially pointing intervals having one endpoint in $\partial\mathbb{B}^5$. In §III.3, we show how individual maps in $\mathcal{R}$ can be weakly approximated by smooth maps by proving:

Theorem III.1 (Singularity Cancellation) *If $u \in \mathcal{R}$, Γ is a minimal $\mathbb{Z}_2$ connection for $\mathrm{Sing}\, v$, and $\varepsilon > 0$, then there exists a smooth $u_\varepsilon \in W^{2,2}(\mathbb{B}^5, \mathbb{S}^3) \cap \mathcal{C}^\infty$ so that $u_\varepsilon(x) = u(x)$ whenever $dist(x, \Gamma) > \varepsilon$ and*

$$\int_{\mathbb{B}^5} |\nabla^2 u_\varepsilon|^2 \, dx \ \le\ \varepsilon \ +\ \int_{\mathbb{B}^5} |\nabla^2 u|^2 \, dx \ +\ c_{\mathbb{SH}} \mathcal{H}^1(\Gamma)$$

where $c_{\mathbb{SH}} \ =\ \int_{\mathbb{S}^4} \left|\nabla^2_{tan}(\mathbb{SH})\right|^2 \, d\mathcal{H}^4 \ <\ \infty.$
See Remark III.1 concerning this constant.

The core of our work, however, involves proving:

Theorem IV.2 (Length Bound) *For any $u \in \mathcal{R}$, $\mathrm{Sing}\, u$ has a $\mathbb{Z}_2$ connection Γ satisfying*

$$\mathcal{H}^1(\Gamma) \ \le\ c \int_{\mathbb{B}^5} |\nabla^2 u|^2 \, dx \ ,$$

for some absolute constant c.

Combining this length bound with Lemma III.2 and Theorem III.1, we readily establish, in Section V, that any Sobolev map in $W^{2,2}(\mathbb{B}^5, \mathbb{S}^3)$ has a $W^{2,2}$ weak approximation by a sequence of smooth maps.

We prove the length bound in Section IV by finding a suitable connection Γ through three applications of the coarea formula. For a regular value $p \in \mathbb{S}^3$ for u, the fiber $\Sigma = u^{-1}\{p\}$ is a smooth surface with cone point singularities at $\mathrm{Sing}\, u$. By the coarea formula, we may choose this p so that

$$\int_{u^{-1}\{p\}} \frac{|\nabla u|^4 + |\nabla^2 u|^2}{J_3 u} \, d\mathcal{H}^2 \ . \tag{I.2}$$

Then we need to choose connectiong curves on Σ. To do this we choose an orthonormal frame $\tilde{\tau}_1$, $\tilde{\tau}_2$, $\tilde{\tau}_3$ of the normal bundle of the surface $\Sigma = u^{-1}\{p\}$ by ortho-normalizing the v pull-backs of a basis of $\mathrm{Tan}\,(\mathbb{S}^3, p)$. Inequality (I.2) gives that

$$\int_\Sigma |\nabla \tilde{\tau}_j| \, d\mathcal{H}^2 \ \le\ c \int_{\mathbb{B}^5} |\nabla^2 u|^2 \, dx \ .$$

We show how a a.e. oriented 2 plane in $\mathbb{R}^5$ determines at every point $x \in \Sigma$, with a finite exceptional set $b_1, \ldots, b_j$, an orthogonal basis of $\mathrm{Nor}(\Sigma, x)$, thought of as a reference normal framing. There is a unique $\gamma(x) \in \mathbb{SO}(3) \simeq \mathbb{RP}^3$ and one gets some curves on Σ, with total length bounded by a multiple of $c \int_{\mathbb{B}^5} |\nabla^2 u|^2 dx$ by choosing $\gamma^{-1}(E)$ where E is a suitable great $\mathbb{RP}^2 \subset \mathbb{SO}(3)$. The curves starting at the some a_i may end in either another a_k or in $\partial\mathbb{B}^5$ or (unfortunately) in a point b_ℓ where the reference framing degenerates. More argument, including another use of the coarea formula is required in sections IV.8, IV.9 to find additional curves of controlled length connecting b_ℓ to another b_m or to $\partial\mathbb{B}^5$. Putting all these curves together gives a $\mathbb{Z}_2$ connection for $\mathrm{Sing}\, u$ satisfying the desired length bound.

II Preliminaries

We will let c denote an absolute constant whose value may change from statement to statement and which is usually easily estimable. Here for $0 \le k \le m$ and various k dimensional subsets A of $\mathbb{R}^m$,

$$\int_A f \, d\mathcal{H}^k \ =\ \int_A f(y) \, d\mathcal{H}^k y$$

will denote integration with respect to k dimensional Hausdorff measure. However, in top dimension where $\mathcal{H}^m$ coincides with Lebesgue measure, we will use the use the standard notations $\int_A f \, dx \ =\ \int_A f(x) \, dx$.

Lemma II.1 *For each positive integer m, there is a positive constant c_m so that*

$$\|v\|^2_{W^{2,2}(\mathbb{B}^m,N)} \;\leq\; c_m \left[\, (\operatorname{diam} N)^2 \;+\; \int_{\mathbb{B}^m} |\nabla^2 v|^2 \, dx \,\right]$$

for any compact Riemannian submanifold N of $\mathbb{R}^\ell$ and $v \in W^{2,2}(\mathbb{B}^m, N)$.

Proof. Here $\|v\|^2_{W^{2,2}(\mathbb{B}^m,N)} \;=\; \int_{\mathbb{B}^m} \left(|v|^2 + |\nabla v|^2 + |\nabla^2 v|^2 \right) dx$. We clearly have the estimate

$$\int_{\mathbb{B}^m} |v|^2 dx \leq \; \mathcal{H}^m(\mathbb{B}^m)(\operatorname{diam} N)^2 \; .$$

Moreover, by the Poincaré inequality,

$$\begin{aligned} \int_{\mathbb{B}^m} |\nabla v|^2 \, dx \quad &= \quad \sum_{i=1}^m \int_{\mathbb{B}^m} \left| \frac{\partial v}{\partial x_i} \right|^2 dx \leq \sum_{i=1}^m 2 \int_{\mathbb{B}^m} \left| \frac{\partial v}{\partial x_i} - \left(\frac{\partial v}{\partial x_i} \right)_{avg} \right|^2 dx \;+\; 2 \int_{\mathbb{B}^m} \left| \left(\frac{\partial v}{\partial x_i} \right)_{avg} \right|^2 dx \\ &\leq \quad 2m\mathbf{C}_{\mathbb{B}^m} \int_{\mathbb{B}^m} |\nabla^2 v|^2 \, dx \;+\; \sum_{i=1}^m 2\mathcal{H}^m(\mathbb{B}^m) \left| \left(\frac{\partial v}{\partial x_i} \right)_{avg} \right|^2 \; . \end{aligned}$$

It only remains to bound $\left(\frac{\partial v}{\partial x_i} \right)_{avg}$. We will do the case $i = 1$, the cases $i \geq 2$ being similar. By Fubini's theorem and the absolutely continuity of v on a.e. line in the $(1, 0, \ldots, 0)$ direction,

$$\begin{aligned} &\mathcal{H}^m(\mathbb{B}^m) \left| \left(\frac{\partial v}{\partial x_1} \right)_{avg} \right| \;=\; \left| \int_{\mathbb{B}^m} \frac{\partial v}{\partial x_1} \, dx \right| \;=\; \left| \int_{\mathbb{B}^{m-1}} \int_{-\sqrt{1-|y|^2}}^{\sqrt{1-|y|^2}} \frac{\partial v}{\partial x_1}(t, y_1, \ldots, y_{m-1}) \, dt \, dy \right| \\ &\leq \; \int_{\mathbb{B}^{m-1}} \left| v(\sqrt{1-|y|^2}, y_1, \ldots, y_{m-1}) - v(-\sqrt{1-|y|^2}, y_1, \ldots, y_{m-1}) \right| \, dy \;\leq\; \mathcal{H}^{m-1}(\mathbb{B}^{m-1}) \operatorname{diam} N \; . \end{aligned}$$

■

A Formula for the Suspension of the Hopf Map

Let $\mathbb{H} : \mathbb{S}^3 \to \mathbb{S}^2$ denote the standard Hopf map [HR] :

$$\mathbb{H}(x_1, x_2, x_3, x_4) \;=\; \left(2x_1x_2 + 2x_3x_4 \, , \, 2x_1x_4 - 2x_2x_3 \, , \, x_1^2 + x_3^2 - x_2^2 - x_4^2 \right)$$

and $\mathbb{SH} : \mathbb{S}^4 \to \mathbb{S}^3$ be its suspension:

$$\mathbb{SH}(x_0, x_1, \cdots, x_4) \;=\; \left(x_0 \; , \; \sqrt{1 - x_0^2} \cdot \mathbb{H} \left(\frac{x_1}{\sqrt{x_1^2 + \cdots + x_4^2}} \, , \, \ldots \, , \, \frac{x_4}{\sqrt{x_1^2 + \cdots + x_4^2}} \right) \right) \; .$$

The latter map generates the nonzero element of $\Pi_4(\mathbb{S}^3) \simeq \mathbb{Z}_2$. Also, its homogeneous degree 0 extension

$$\mathbb{SH}(x/|x|) \;\in\; W^{2,2}(\mathbb{B}^5, \mathbb{S}^3) \; .$$

In particular, $\int_{\mathbb{B}^5} \left| \nabla^2 \left(\mathbb{SH}(x/|x|) \right) \right|^2 dx \;= c_{\mathbb{SH}}$ where

$$c_{\mathbb{SH}} \;=\; \int_{\mathbb{S}^4} \left| \nabla^2_{tan}(\mathbb{SH}) \right|^2 \, d\mathcal{H}^4 \;<\; \infty \; . \tag{II.3}$$

While the explicit formula for a suspension of the Hopf map is handy for simplifying proofs, the constant $c_{\mathbb{SH}}$, which occurs in the conclusion of Theorem III.1 can, by Remark III.1, be replaced by a more natural constant.

III A Strongly Dense Family with Isolated Singularities

Let $\mathcal{R}$ denote the class of $W^{2,2}(\mathbb{B}^5,\mathbb{S}^3)$ maps that are smooth except for finitely many suspension Hopf singularities. That is,

$$u\in\mathcal{R}\quad\Longleftrightarrow$$
$$u\in\mathcal{C}^\infty(\mathbb{B}^5\setminus\{a_1,\dots,a_m\},\mathbb{S}^3)\quad\text{and}\quad u(x)\;=\;\mathbb{SH}\left(\frac{x-a_i}{|x-a_i|}\right)\quad\text{on}\quad \mathbb{B}_{\delta_0}(a_i)\setminus\{a_i\}\tag{III.4}$$

for some finite subset $\{a_1,\dots,a_m\}$ of $\mathbb{B}^5$ and some positive $\delta_0<\min_i\{1-|a_i|,\min_{j\neq i}|a_i-a_j|/2\}$ }.

III.1 Strong Approximation by Maps in $\mathcal{R}$

Lemma III.2 *$\mathcal{R}$ is $W^{2,2}$ strongly dense in $W^{2,2}(\mathbb{B}^5,\mathbb{S}^3)$.*

Proof. Theorem 5 of [BPV] gives the $W^{2,2}$ strongly density of the family $\mathcal{R}_0^{2,2}(\mathbb{B}^5,\mathbb{S}^3)$ of maps $v\in W^{2,2}(\mathbb{B}^5,\mathbb{S}^3)$ which are smooth except for a finite singular set $\{a_1,\dots,a_m\}$ and which satisfy

$$\limsup_{x\to a_i}\big(\;|x-a_i||\nabla v(x)|\;+\;|x-a_i|^2|\nabla^2v(x)|\;\big)\;<\;\infty\;,$$

for $i=1,\dots,m$. Thus, it suffices to show:

For each $v\in\mathcal{R}_0^{2,2}(\mathbb{B}^5,\mathbb{S}^3)$ and $\varepsilon>0$, there is a map $u\in\mathcal{R}$ so that $\|u-v\|^2_{W^{2,2}}<\varepsilon$.

Assuming $\operatorname{Sing} v=\{a_1,\dots,a_m\}$, we will obtain u by modifying v near each point a_i. First fix a positive $\eta<\frac12\min\{\min_{i\neq j}|a_i-a_j|,\min_k(1-|a_k|)\}$ so that

$$L\;=\;\max_i\sup_{0<|x-a_i|<\eta}\big(|x-a_i||\nabla v(x)|\;+\;|x-a_i|^2|\nabla^2v(x)|\big)\;<\;\infty\tag{III.5}$$

We will proceed in two stages: First we will find a positive $\delta<\eta$ depending only on L and then define a map $w\in W^{2,2}(\mathbb{B}^5,\mathbb{S}^3)$ so that

$$w\equiv v\text{ on }\mathbb{B}^5\setminus\cup_{i=1}^m\mathbb{B}_\delta(a_i)\;,\tag{III.6}$$

and, on each ball $\mathbb{B}_{\delta/2}(a_i)$, w is degree-zero homogeneous about a_i, i.e.

$$w(x)\;=\;w\left(a_i+\frac{x-a_i}{|x-a_i|}\right)\quad\text{for}\quad 0<|x-a_i|<\frac12\delta\;.$$

Second, we find a positive $\delta_0<<\frac12\delta$, depending on $w|\cup_{i=1}^m\partial\mathbb{B}_{\delta/2}(a_i)$, and a map $u\in\mathcal{R}$ with $u\equiv w$ on $\mathbb{B}^5\setminus\cup_{i=1}^m\mathbb{B}_{\delta_0}(a_i)$ and, on each ball $\mathbb{B}_{\delta_0}(a_i)$, $u\equiv\mathbb{SH}\left(\frac{x-a_i}{|x-a_i|}\right)$.

For the first step, we first fix a smooth monotone increasing $\lambda:[0,\infty)\to[\frac12,\infty)$ so that

$$\lambda(t)\;=\;\begin{cases}1/2 & \text{for}\quad 0\le t\le\frac12\\ t & \text{for}\quad t\ge1\;.\end{cases}$$

Consider the unscaled situation of a map $V\in\mathcal{C}^\infty(\overline{\mathbb{B}^5}\setminus\mathbb{B}_{\frac12}(0),\mathbb{S}^3)$ with $|\nabla V|+|\nabla^2V|\le L$. Then we define the reparameterized map

$$W(x)=V\left(\lambda(|x|)x\right)\;,$$

and see that, with respect to the radial variable $\rho=|x|$,

$$\frac{\partial W}{\partial\rho}\;\equiv\;\frac{\partial V}{\partial\rho}\quad\text{on}\quad\partial\mathbb{B}^5\;,\qquad\frac{\partial W}{\partial\rho}\;\equiv\;0\quad\text{on}\quad\overline{\mathbb{B}_{\frac12(0)}}\setminus\{0\}\;,$$

Moreover, using explicit pointwise bounds for $|\lambda'|$ and $|\lambda''|$, we readily find an explicit constant C so that

$$|\nabla W(x)| + |\nabla^2 W(x)| \ \le\ C\,L \quad \text{for} \quad \frac{1}{2} \le |x| \le 1\ ,$$

$$|x||\nabla W(x)| + |x|^2|\nabla^2 W(x)| \ \le\ C\,L \quad \text{for} \quad 0 < |x| < \frac{1}{2}\ \ .$$

Now we return to the original scale by defining w to satisfy (III.6) and to have, for each point $x \in \mathbb{B}_\delta(a_i)$,

$$w(x)\ =\ W\left(\frac{x-a_i}{\delta}\right) \quad \text{where} \quad V(x)\ =\ v(a_i+\delta x)\ .$$

Then w belongs to $W^{2,2}(\mathbb{B}^5,\mathbb{S}^3)$ and satisfies the estimate

$$\max_i \sup_{0<|x-a_i|<\delta} \big(|x-a_i||\nabla w(x)|\ +\ |x-a_i|^2|\nabla^2 w(x)|\big)\ \le\ CL\ ,$$

hence,

$$\begin{aligned}\|w-v\|^2_{W^{2,2}} \ &=\ \textstyle\sum_{i=1}^m \int_{\mathbb{B}_\delta(a_i)} \big(|w-v|^2 + |\nabla w - \nabla v|^2 + |\nabla^2 w - \nabla^2 v|^2\big)\,dx \\ &\le\ 2\textstyle\sum_{i=1}^m \int_{\mathbb{B}_\delta(a_i)} \big(|w|^2 + |v|^2 + |\nabla w|^2 + \nabla v|^2 + |\nabla^2 w|^2 + |\nabla^2 v|^2\big)\,dx\ \le\ c(1+L)^2\delta\ .\end{aligned}$$

So we easily choose δ so that $\|w-v\|^2_{W^{2,2}} < \frac{1}{4}\varepsilon$.

For the second step we note that any continuous homotopy between smooth maps of smooth manifolds can be made smooth; hence :

A smooth map $\phi : \mathbb{S}^4 \to \mathbb{S}^3$ *is smoothly homotopic* $\begin{cases} \textit{either to a constant} & \textit{in case } [\![\phi]\!] = 0 \in \Pi_4(\mathbb{S}^4,\mathbb{S}^3) \\ \textit{or to} \quad [\![\mathbb{SH}]\!] & \textit{in case } [\![\phi]\!] \ne 0 \in \Pi_4(\mathbb{S}^4,\mathbb{S}^3)). \end{cases}$

For each $i = 1,\dots,m$, we apply this to the map $\phi_i(x) = w(a_i + \frac{1}{2}\delta x)$ to obtain a smooth homotopy $h_i : [0,1]\times\mathbb{S}^4 \to \mathbb{S}^3$ which connects ϕ_i to $\mathbb{SH}$. Reparameterizing the time variable near 0 and 1, we may assume

$$h_i(t,y)\ =\ \begin{cases} \phi_i(y) & \text{for } t \text{ near } 0 \\ (\mathbb{SH})(y) & \text{for } t \text{ near } 1\ . \end{cases}$$

By smoothness, $K\ =\ \sup_i \|h_i\|_{W^{2,2}} < \infty$, and we will, for some $\delta_0 << \frac{1}{2}\delta$, define the map u by

$$u \equiv w \quad \text{on} \quad \mathbb{B}^5 \setminus \cup_{i=1}^m \mathbb{B}_{\delta_0}(a_i)\ ,$$

$$u(x)\ =\ h_i\left(2 - 2\frac{|x-a_i|}{\delta_0}\ ,\ \frac{x-a_i}{\delta_0}\right) \quad \text{for} \quad x \in \mathbb{B}_{\delta_0}(a_i)\setminus \mathbb{B}_{\frac{1}{2}\delta_0}(a_i)\ ,$$

$$u(x)\ =\ \mathbb{SH}\left(\frac{x-a_i}{|x-a_i|}\right) \quad \text{for} \quad x \in \mathbb{B}_{\frac{1}{2}\delta_0}(a_i)\setminus\{0\}\ ,$$

for $i = 1,\dots,m$. One readily checks that $u \in \mathcal{R}$. Moreover, as in Step 1, we find that

$$\|u-w\|^2_{W^{2,2}}\ = \sum_{i=1}^m \int_{\mathbb{B}_\delta(a_i)} \big(|u-w|^2 + |\nabla u - \nabla w|^2 + |\nabla^2 u - \nabla^2 w|^2\big)\,dx\ \le\ c(1+K)^4\delta_0\ .$$

So we easily choose δ_0 small enough so that $\|u-w\|^2_{W^{2,2}} < \frac{1}{4}\varepsilon$ and obtain the desired estimate

$$\|u-v\|^2_{W^{2,2}}\ \le 2\|u-w\|^2_{W^{2,2}}\ +\ 2\|w-v\|^2_{W^{2,2}}\ <\ \varepsilon\ .$$

■

III.2 Insertion of an $\mathbb{SH}$ Bubble into a Map from $\mathbb{B}^4$ to $\mathbb{S}^3$

Arguing as in the proof of Lemma III.2, we first fix a monotone increasing smooth function μ on $[0,\infty)$ so that

$$\mu(t) \ = \ \begin{cases} 0 & \text{for} \quad 0 \le t \le \frac{1}{2} \\ t & \text{for} \quad t \ge 1 \ , \end{cases}$$

$|\mu'| \le 3$, and $|\mu''| \le 16$. We readily prove the following:

Lemma III.3 (Initial Reparameterization) *There an absolute constant C so that, for any smooth $f : \mathbb{B}^4 \to \mathbb{S}^3$ and $0 < \sigma < 1$, the map*

$$f_\sigma : \mathbb{B}^4 \to \mathbb{S}^3 \ , \quad f_\sigma(y) \ = \ f\left(\mu\left(\sigma^{-1}|y|\right)|y|^{-1}\sigma y\right) \quad \text{for } y \in \mathbb{B}^4 \ ,$$

coincides with f on $\mathbb{B}^4 \setminus \mathbb{B}^4_\sigma$, is identically equal to $f(0)$ on $\mathbb{B}^4_{\sigma/2}$, and satisfies

$$\sup_{\mathbb{B}^4_\sigma} |\nabla f_\sigma| \ \le \ C \sup_{\mathbb{B}^4_\sigma} |\nabla f_\sigma| \ , \qquad \sup_{\mathbb{B}^4_\sigma} |\nabla^2 f_\sigma| \ \le \ C\sigma^{-1} \sup_{\mathbb{B}^4_\sigma} |\nabla^2 f| \ .$$

In particular,

$$\int_{\mathbb{B}^4_\sigma} |\nabla^2 f_\sigma|^2 \, dx \ \le \ C^2 \mathcal{H}^4(\mathbb{B}^4) \sup_{\mathbb{B}^4_\sigma} |\nabla^2 f|^2 \, \sigma^2 \ . \tag{III.7}$$

Construction of an $\mathbb{SH}$ Bubble

Recalling that $\mathbb{SH}(0,1,0,0,0) = (0,0,0,1)$, we will first slightly modify $\mathbb{SH}$ to be constant near $(0,1,0,0,1)$. Consider the spherical coordinate parameterization,

$$\Upsilon : [0,\pi] \times \mathbb{S}^3 \to \mathbb{S}^4 \ , \quad \Upsilon(\rho,\omega) = ((\sin\rho)\omega_1, \cos\rho, (\sin\rho)\omega_2, (\sin\rho)\omega_3, (\sin\rho)\omega_4) \ .$$

Arguing again as in the proofs of Lemma III.2 and Lemma III.3, we let

$$M_{\rho_0}(\rho,\omega) \ = \ \left(\mu(\rho_0^{-1}\rho)\rho_0\rho \, , \omega\right) \quad \text{for} \quad 0 \le \rho \le \rho_0 << 1 \quad \text{and} \quad \omega \in \mathbb{S}^3 \ ,$$

and define $\Phi_{\rho_0} : \mathbb{S}^4 \to \mathbb{S}^4$ by

$$\Phi_{\rho_0}(y) \ = \ \begin{cases} \Upsilon \circ M_{\rho_0} \circ \Upsilon^{-1}\{y\} & \text{for } y \in \Upsilon\left([0,\rho_0] \times \mathbb{S}^4\right) \\ \text{y} & \text{otherwise} \ . \end{cases}$$

Then Φ_{ρ_0} is surjective, and maps the entire spherical cap

$$\Omega_{\rho_0/2} \ = \ \Upsilon\left([0,\rho_0/2) \times \mathbb{S}^4\right) \ = \ \mathbb{S}^4 \cap \mathbb{B}^5_{2\sin(\rho_0/4)}\left((0,1,0,0,0)\right)$$

to its center point $(0,1,0,0,0)$.

We now consider the composition $\mathbb{SH} \circ \Phi_{\rho_0}$ The homotopy class is unchanged

$$[\![\mathbb{SH} \circ \Phi_{\rho_0}]\!] \ = \ [\![\mathbb{SH}]\!] \ \ne 0 \ \in \ \Pi_4(\mathbb{S}^3) \ . \tag{III.8}$$

Noting that $\rho_0 \frac{\partial^2 M_{\rho_0}}{\partial \rho^2}$ is bounded independent of ρ_0 and has support in $\Upsilon\left([0,\rho_0] \times \mathbb{S}^3\right)$, we readily verify, as in (III.7), that

$$\int_{\mathbb{S}^4} |\nabla^2_{tan}(\mathbb{SH}\circ\Phi_{\rho_0})|^2 d\mathcal{H}^4 \ = \ \int_{\mathbb{S}^4 \setminus \Omega_{\rho_0}} |\nabla^2_{tan}(\mathbb{SH})|^2 d\mathcal{H}^4 + \int_{\Omega_{\rho_0}} |\nabla^2_{tan}(\mathbb{SH}\circ\Phi_{\rho_0})|^2 d\mathcal{H}^4 \to \quad c_{\mathbb{SH}} + 0 \ , \tag{III.9}$$

as $\rho_0 \to 0$.

Since the stereographic projection

$$\Pi : \mathbb{S}^4 \setminus \{(0,1,0,0,0)\} \to \mathbb{R}^4 \ , \quad \Pi(x_0,x_1,x_2,x_3,x_4) \ = \ \left(\frac{x_0}{1-x_1}, \frac{x_2}{1-x_1}, \frac{x_3}{1-x_1}, \frac{x_4}{1-x_1}\right)$$

is conformal, we get the conformal diffeomorphism

$$\Lambda_{\rho_0} \ : \ \overline{\mathbb{B}^4} \ \to \ \mathbb{S}^4 \setminus \Omega_{\rho_0/2} \ , \quad \Lambda_{\rho_0}(y) \ = \ \Pi^{-1}\left[\left(\frac{\sin(\rho_0/2)}{1-\cos(\rho_0/2)}\right) y\right] \ .$$

We see that

$$\mathbb{SH} \circ \Phi_{\rho_0} \circ \Lambda_{\rho_0} \ : \ \overline{\mathbb{B}^4} \ \to \ \mathbb{S}^3$$

is a smooth surjection which sends $\partial\mathbb{B}^4$ identically to $\mathbb{SH}(0,1,0,0,0) = (0,0,0,1)$.

We wish to have a similar map that has boundary values being a a possibly different constant $\xi \in \mathbb{S}^3$. To get a suitable formula, it will be handy to recall that $\mathbb{S}^3$, being identified with the unit quaternions,

$$\{\, x_1 + x_2\mathbf{i} + x_3\mathbf{j} + x_4\mathbf{k} \ : \ x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1 \,\}$$

has a product $\star$ and, in particular, that

$$(0,0,0,-1) \star (0,0,0,1) \ = \ (-\mathbf{k}) \star \mathbf{k} \ = \mathbf{1} \ = \ (1,0,0,0) \ .$$

So now we define, for every $\rho_0 > 0$ and $\xi \in \mathbb{S}^3$, the bubble

$$\mathcal{B}_{\rho_0,\xi} \ : \ \mathbb{B}^4 \ \to \ \mathbb{S}^3 \ , \quad \mathcal{B}_{\rho_0,\xi} \ \equiv \ \xi \star (-\mathbf{k}) \star \mathbb{SH} \circ \Phi_{\rho_0} \circ \Lambda_{\rho_0} \ , \tag{III.10}$$

which is identically equal to ξ on $\partial\mathbb{B}^4$. By the conformal invariance of the Hessian energy and (III.9),

$$\int_{\mathbb{B}^4} |\nabla^2 \mathcal{B}_{\rho_0,\xi}|^2 \, dy \ = \ \int_{\mathbb{S}^4} |\nabla^2_{tan}(\mathbb{SH} \circ \Phi_{\rho_0})|^2 \, d\mathcal{H}^4 \ \to \ \int_{\mathbb{S}^4} |\nabla^2_{tan}(\mathbb{SH})|^2 \, d\mathcal{H}^4 \ = \ c_{\mathbb{SH}} \qquad \text{as } \rho_0 \to 0 \ . \tag{III.11}$$

Now, for a smooth map $f : \mathbb{B}^4 \to \mathbb{S}^3$, $0 < \sigma < 1$, and $0 < \rho_0 << 1$, we define the "bubbled" map $f_{\sigma,\rho_0} : \mathbb{B}^4 \to \mathbb{S}^3$,

$$f_{\sigma,\rho_0}(y) \ = \ \begin{cases} f_\sigma(y) & \text{for } \sigma/2 < |y| < 1 \ , \\ \mathcal{B}_{\rho_0,f(0)}(2y/\sigma) & \text{for } |y| \le \sigma/2 \ . \end{cases} \tag{III.12}$$

Note that f_{σ,ρ_0} is continuous because both expressions equal $f(0)$ on $\partial\mathbb{B}^4_{\sigma/2}$. Moreover, all the positive order derivatives of both expressions vanish here because the smooth map f_σ is constant on $\mathbb{B}^4_{\sigma/2}$ while the smooth reparameterization Φ_{ρ_0} is constant on $\Omega_{\rho_0/2}$. So the map f_{σ,ρ_0} is, in fact, smooth. Moreover, since the Hessian energy is invariant under change of scale,

$$\begin{aligned} \int_{\mathbb{B}^4} \left|\nabla^2 f_{\sigma,\rho_0}\right|^2 dy \quad &= \quad \int_{\mathbb{B}^4 \setminus \mathbb{B}^4_{\sigma/2}} \left|\nabla^2 f_\sigma\right|^2 dy \ + \ \int_{\mathbb{B}^4_{\sigma/2}} \left|\nabla^2 \left[\mathcal{B}_{\rho_0,f(0)}\left(\tfrac{2}{\sigma}(\cdot)\right)\right]\right|^2 dy \\ &= \quad \int_{\mathbb{B}^4 \setminus \mathbb{B}^4_\sigma} \left|\nabla^2 f\right|^2 \, dy \ + \int_{\mathbb{B}^4_\sigma \setminus \mathbb{B}^4_{\sigma/2}} \left|\nabla^2 f_\sigma\right|^2 \, dy \ + \ \int_{\mathbb{B}^4} \left|\nabla^2 \mathcal{B}_{\rho_0,f(0)}\right|^2 \, dy \\ &\to \quad \int_{\mathbb{B}^4} \left|\nabla^2 f\right|^2 \, dy \quad + \quad 0 \quad + \quad \int_{\mathbb{B}^4} \left|\nabla^2 \mathcal{B}_{\rho_0,f(0)}\right|^2 \, dy \qquad \text{as} \quad \sigma \to 0 \ . \end{aligned} \tag{III.13}$$

Note that the middle equation also gives, with (III.7) the bound

$$\int_{\mathbb{B}^4} \left|\nabla^2 f_{\sigma,\rho_0}\right|^2 dy \ \le \ (1 + C^2) \sup_{\mathbb{B}^4} |\nabla^2 f|^2 \ + \int_{\mathbb{B}^4} \left|\nabla^2 \mathcal{B}_{\rho_0,f(0)}\right|^2 \, dy, \tag{III.14}$$

independent of σ.

III.3 Singularity Cancellation of a Map in $\mathcal{R}$

Theorem III.1 *If $u \in \mathcal{R}$, Γ is a minimal $\mathbb{Z}_2$ connection for $\mathrm{Sing}\, v$, and $\varepsilon > 0$, then there exists a smooth $u_\varepsilon \in W^{2,2}(\mathbb{B}^5, \mathbb{S}^3) \cap \mathcal{C}^\infty$ so that $u_\varepsilon(x) = u(x)$ whenever $\mathrm{dist}(x, \Gamma) > \varepsilon$ and*

$$\int_{\mathbb{B}^5} |\nabla^2 u_\varepsilon|^2\, dx \ \le\ \varepsilon\ +\ \int_{\mathbb{B}^5} |\nabla^2 u|^2\, dx\ +\ c_{\mathbb{SH}} \mathcal{H}^1(\Gamma)\ .$$

Proof. Note that, by slightly rescaling near $\partial \mathbb{B}^5$, we may assume that u extends smoothly to a neighborhood of $\overline{\mathbb{B}^5}$.

First, using (III.11), we fix a positive ρ_0 to be small enough so that,

$$\left| c_{\mathbb{SH}}\ -\ \int_{\mathbb{B}^4} |\nabla^2 \mathcal{B}_{\rho_0,\xi}|^2\, dy \right|\ <\ \frac{\varepsilon}{4(1+\mathcal{H}^1(\Gamma))} \tag{III.15}$$

for all $\xi \in \mathbb{S}^3$. Also we recall that a minimal $\mathbb{Z}_2$ connection for u is the union of a finite family $\mathcal{I}$ of disjoint closed intervals $I = [a_I, b_I]$ where

$$a_I\ \in\ \mathrm{Sing}\, u \quad \text{and} \quad b_I\ \in\ \begin{cases} \textit{either} & \mathrm{Sing}\, u \\ \textit{or} & \partial\mathbb{B}^5 \text{ with } I \perp \partial \mathbb{B}^5\ . \end{cases} \tag{III.16}$$

Second, we fix a positive δ_0 so that:

(1) $\delta_0 < \frac{1}{2} \min_{a \in \mathrm{Sing}\, u} \{1 - |a|, \min_{a \neq \tilde{a} \in \mathrm{Sing}\, u} |a - \tilde{a}|\, \}$

(2) $\delta_0\ < \frac{1}{2} \min\{|x - \tilde{x}|\ :\ x \in I,\ \tilde{x} \in \tilde{I},\ I \neq \tilde{I} \in \mathcal{I}\ \}$.

(3) $u \in \mathcal{C}^\infty(\mathbb{B}^5_{1+\delta_0} \setminus \mathrm{Sing}\, u, \mathbb{S}^3)$.

(4) $\delta_0 < (1 + c_{\mathbb{SH}})^{-1} \left(1 + \mathrm{card}(\mathrm{Sing}\, u)\right)^{-1} \varepsilon/5$

Our main step in constructing u_ε will be to use, for each $I \in \mathcal{I}$, the bubble insertion of §III.2 in each cross-section of a pinched cylindrical region V_I of radius $\delta_0/9$. Near the singular endpoints of I, V_I is pinched to be a round cone with opening angle $2\arctan(\frac{1}{9})$.

To describe the explicit construction, we need some notation. With $I = [a_I, b_I] \in \mathcal{I}$ as above in (III.16), let

$$|I|\ =\ |b_I - a_I|, \quad \mathbf{e}_I\ =\ (b_I - a_I)/|I|\ \in \mathbb{S}^4, \qquad \Pi_I : \mathbb{R}^5 \to \mathbb{R}, \quad \Pi_I(x) = x \cdot \mathbf{e}_I - a_I \cdot \mathbf{e}_I\ ,$$

and $B_I(t, r)$ be the open ball in the 4 dimensional affine plane $\Pi_I^{-1}\{t\}$ with center $a_I + t\mathbf{e}_I$ and radius r. We now define V_I to be the pinched cylindrical region

$$V_I\ =\ \bigcup_{0<t<|I|} B_I\left(t, r_I(t)\right)\ ,$$

by using a fixed smooth function $\nu : [0, \infty) \to [0, 1]$ with

$$\nu(t)\ =\ \begin{cases} t & \text{for } 0 \le t \le \frac{1}{2} \\ 1 & \text{for } 1 \le t\ , \end{cases}$$

to define the smooth radius function

$$r_I(t)\ =\ \begin{cases} (\delta_0/9)t & \text{for } 0 \le t \le \frac{1}{2}\delta_0 \\ (\delta_0/9)\nu(t/\delta_0) & \text{for } \frac{1}{2}\delta_0 \le t \le \delta_0 \\ \delta_0/9 & \text{for } \delta_0 \le t \le |I| - \delta_0 \\ (\delta_0/9)\nu((|I| - t)/\delta_0) & \text{in case } b_I \in \mathrm{Sing}\, u \text{ and } |I| - \delta_0 \le t \le |I| - \frac{1}{2}\delta_0 \\ (\delta_0/9)\,(|I| - t) & \text{in case } b_I \in \mathrm{Sing}\, u \text{ and } |I| - \frac{1}{2}\delta_0 \le t \le |I| \\ \delta_0/9 & \text{in case } b_I \in \partial\mathbb{B}^5 \text{ and } |I| - \delta_0 \le t \le |I|\ . \end{cases}$$

Thus ∂V_I is a smooth hypersurface except for the conepoint(s) a_I (and b_I in case $b_I \in \operatorname{Sing} u$).

For convenience, we fix an orthonormal basis $\mathbf{e}_1^I, \ldots, \mathbf{e}_4^I$ for the orthogonal complement of $\mathbb{R}\mathbf{e}_I$. For each $t \in \mathbb{R}$, we will use the affine similarity

$$A_{I,t} : \mathbb{R}^4 \to \mathbb{R}^5 \ , \quad A_{I,t}(y_1, \ldots, y_4) \ = \ a_I + t\mathbf{e}_I + r_I(t)\left[y_1\mathbf{e}_1^I + \cdots + y_4\mathbf{e}_4^I\right]$$

so that $A_{I,t}(\mathbb{B}^4) \ = \ B_I\left(t, r_I(t)\right)$.

For $0 < \sigma < 1$, we recall (III.12) and define the smooth reparameterized map

$$v_\sigma(x) \ = \begin{cases} u(x) & \text{for } x \in \mathbb{B}^5 \setminus \bigcup_{I\in\mathcal{I}} V_I \cup \{a_I\} \cup \{b_I\} \\ (u \circ A_{I,t})_{\sigma,\rho_0}\left(A_{I,t}^{-1}(x)\right) & \text{for } x \in B_I\left(t, r_I(t)\right) \ . \end{cases}$$

Observe that v_σ actually coincides with $u(x)$ outside the "σ thin" set $\bigcup_{I\in\mathcal{I}} V_I^\sigma \cup \{a_I\} \cup \{b_I\}$ where

$$V_I^\sigma \ = \ \bigcup_{0<t<|I|} B_I\left(t, \sigma r_I(t)\right) \ .$$

The explicit formulas given above and in the earlier parts of §III.3 show the qualitative smoothness of v_σ on $\mathbb{B}^5 \setminus \operatorname{Sing} u$. Our next goal is to verify that

$$\limsup_{\sigma\to 0} \sum_{I\in\mathcal{I}} \int_{V_I^\sigma} \left|\nabla^2 v_\sigma\right|^2 dx \quad < \quad c_{\mathbb{SH}}\mathcal{H}(\Gamma) \ + \ \frac{\varepsilon}{2} \ . \tag{III.17}$$

We define

$$J_I \ = \ \left\{ t \ : \ \frac{1}{2}\delta_0 \ \le \ t \ \le \ \left\{ \begin{array}{ll} |I| - \frac{1}{2}\delta_0 & \text{in case } b_I \in \operatorname{Sing} u \\ |I| & \text{in case } b_I \in \partial\mathbb{B}^5 \ , \end{array} \right\} \right.$$

and note that on J_I, the scaling factor $r_I(t)$ satisfies $\frac{1}{18}\delta \ \le \ r_I(t) \ \le \ \frac{1}{9}\delta_0$ while $|r_I'(t)|$ and $|r_I''(t)|$ are bounded. We will use the corresponding truncated sets

$$W_I \equiv V_I \cap \Pi_I^{-1}(J_I) \ = \ \bigcup_{t\in J_I} B_I\left(t, r_I(t)\right) \ , \quad W_I^\sigma \equiv V_I^\sigma \cap \Pi_I^{-1}(J_I) \ = \ \bigcup_{t\in J_I} B_I\left(t, \sigma r_I(t)\right) \ .$$

We have the pointwise bound

$$L \ = \ \sup_{W_I} \left(|\nabla u|^2 + |\nabla^2 u|^2\right) \ < \ \infty \ ; \tag{III.18}$$

hence, by (III.14),

$$\sup_{t\in J_I} \sup_{0<\sigma<1} \int_{\mathbb{B}^4} \left|\nabla^2 (u \circ A_{I,t})_{\sigma,\rho_0}\right|^2 dy \ < \ \infty \ .$$

Note that the orthogonality of the five vectors $\mathbf{e}_I, \mathbf{e}_1^I, \cdots, \mathbf{e}_4^I$ lead to the decomposition of the squared Hessian norm into pure second partial derivatives

$$|\nabla^2(\cdot)|^2 \quad = \quad |\nabla_{\mathbf{e}_I,\mathbf{e}_I}(\cdot)|^2 \ + \ \left|(\nabla_{\mathbf{e}_1^I,\mathbf{e}_1^I}(\cdot)\right|^2 \ + \ \cdots \ + \ \left|(\nabla_{\mathbf{e}_4^I,\mathbf{e}_4^I}(\cdot)\right|^2 \ ,$$

which we will abbreviate as $\left|\nabla^2_{\mathbf{e}_I}(\cdot)\right|^2 \ + \ \left|\nabla^2_{\mathbf{e}_I^\perp}(\cdot)\right|^2$.

It follows from Fubini's Theorem, the conformal invariance of the 4 dimensional Hessian energy, (III.13), dominated convergence, and (III.15) that

$$\begin{aligned} \int_{W_I^\sigma} \left|\nabla^2_{\mathbf{e}_I^\perp} v_\sigma\right|^2 dx \quad &= \quad \int_{t\in J_I} \int_{B_I(t,r_I(t))} \left|\nabla^2_{\mathbf{e}_I^\perp} v_\sigma\right|^2 d\mathcal{H}^4\, dt \\ &= \quad \int_{t\in J_I} \int_{\mathbb{B}^4} \left|\nabla^2 \left(u \circ A_{I,t}\right)_{\sigma,\rho_0}\right|^2 dy\, dt \\ &\to \quad \int_{t\in J_I} \left[\ 0 \ + \int_{\mathbb{B}^4} \left|\nabla^2 \mathcal{B}_{\rho_0,u(a_I+t\mathbf{e}_I)}\right|^2 dy\right] dt \ \le \ c_{\mathbb{SH}}|I| \ + \ \tfrac{\varepsilon|I|}{4(1+\mathcal{H}^1(\Gamma))} \ , \end{aligned} \tag{III.19}$$

as $\sigma \to 0$. Thus

$$\limsup_{\sigma\to 0} \sum_{I\in\mathcal{I}} \int_{W_I^\sigma} \left|\nabla^2_{\mathbf{e}_I^\perp} v_\sigma\right|^2 \, dx \quad \le \quad \sum_{I\in\mathcal{I}} \left[c_{\mathbb{SH}} |I| + \frac{\varepsilon |I|}{4(1+\mathcal{H}^1(\Gamma))} \right] \quad \le \quad c_{\mathbb{SH}} \mathcal{H}^1(\Gamma) \; + \; \varepsilon/4 \; . \qquad \text{(III.20)}$$

To get the full squared Hessian integral $\int_{W_I^\sigma} \left|\nabla^2 v_\sigma\right|^2 \, dx$, we also need to consider $\int_{W_I^\sigma} \left|\nabla^2_{\mathbf{e}_I} v_\sigma\right|^2 \, dx$, which involves computing $\frac{\partial^2}{\partial t^2}$ of various terms. To estimate the last integral, it again suffices by Fubini's theorem, Lemma III.3, (III.12), and changing variables to consider

$$\int_{t\in J_I} \left[\int_{\mathbb{B}^4_\sigma \setminus \mathbb{B}^4_{\sigma/2}} \left| \frac{\partial^2}{\partial t^2} (u \circ A_{I,t}) \right|^2 dy \, dt \; + \; \int_{\mathbb{B}^4_{\sigma/2}} \left| \frac{\partial^2}{\partial t^2} \left[\mathcal{B}_{\rho_0, u(a_I + t\mathbf{e}_I)} \left(\frac{2}{\sigma} (\cdot) \right) \right] \right|^2 dy \right] \; dt \; . \qquad \text{(III.21)}$$

The chain rule and the bounds of $|r'|$ and $|r''|$ on J_I give the pointwise bound

$$\left| \frac{\partial^2}{\partial t^2} (u \circ A_{I,t}) \right| \;\; = \;\; \left| \frac{\partial^2}{\partial t^2} \left[u \left(a_I + t\mathbf{e}_I + r_I(t) y \right) \right] \right| \; \le \; cL \; ,$$

and definition (III.10) gives the bound

$$\left| \frac{\partial^2}{\partial t^2} \mathcal{B}_{\rho_0, u(a_I + t\mathbf{e}_I)} \right| \;\; = \;\; \left| \frac{\partial^2}{\partial t^2} \left[u \left(a_I + t\mathbf{e}_I \right) \star (-\mathbf{k}) \star \mathbb{SH} \circ \Phi_{\rho_0} \circ \Lambda_{\rho_0} \; \right] \right| \; \le \; L \; .$$

Integrating implies that (III.21) is bounded by $(c+1)L|I|\mathcal{H}^4(\mathbb{B}^4_\sigma)$, and we deduce that

$$\lim_{\sigma\to 0} \int_{W_I^\sigma} \left|\nabla^2_{\mathbf{e}_I} v_\sigma\right|^2 \, dx \;\; = \;\; 0 \; . \qquad \text{(III.22)}$$

Next we consider the conical end(s) $V_I^\sigma \setminus W_I^\sigma$. By our choice of δ_0, u is degree-0 homogeneous about a_I on the region $\mathbb{B}^5_{\delta_0}(a_I)$. It follows that all of the normalized bubbled functions $(u \circ A_{I,t})_{\sigma,\rho_0}$ coincide for $0 < t \le \sigma/2$. Thus, in the one conical end $V_I^\sigma \cap \Pi_I^{-1}(0, \delta_0/2]$, v_σ is also degree-0 homogeneous about a_I. So we can easily estimate the Hessian integral there by using spherical coordinates about a_I. Note that radial projection of the 4 dimensional Euclidean ball $V_I^\sigma \cap \Pi^{-1}\{\delta_0/2\}$ onto the small spherical cap $V_I^\sigma \cap \partial \mathbb{B}^5_{\delta_0/2}$ is a smooth diffeomorphism with easily computed $\mathcal{C}^2$ bounds on it and its inverse. In particular, we see that, for σ sufficiently small,

$$E_\sigma \equiv \int_{V_I^\sigma \cap \partial \mathbb{B}^5_{\delta_0/2}(a_I)} \left|\nabla^2_{tan} v_\sigma\right|^2 d\mathcal{H}^4 \;\; \le \;\; 2 \int_{\mathbb{B}^4_\sigma} \left| \nabla^2 \left(u \circ A_{I,\delta_0/2} \right)_{\sigma,\rho_0} \right|^2 dy \quad < \quad 2 + 2c_{\mathbb{SH}} \; .$$

By our initial choice (4) of δ_0 we find that, for such σ,

$$\int_{V_I^\sigma \cap \Pi_I^{-1}(0,\delta_0/2]} \left|\nabla^2 v_\sigma\right|^2 \, dx \quad \le \quad (\delta_0/2) E_\sigma \quad \le \quad \left(1 + \mathrm{card}(\mathrm{Sing}\, u)\right)^{-1} \varepsilon/5 \; .$$

In case $b_I \in \mathrm{Sing}\, u$, we make a similar estimate near b_I. In any case, we now have

$$\limsup_{\sigma\to 0} \sum_{I\in\mathcal{I}} \int_{V_I^\sigma \setminus W_I^\sigma} \left|\nabla^2 v_\sigma\right|^2 \, dx \;\; \le \;\; \varepsilon/5 \; ,$$

which together with (III.19) and (III.22), gives the desired Hessian integral estimate (III.17) for v_σ.

Now using (III.17), we are ready to fix a positive $\sigma_0 < 1$ so that

$$\sum_{I\in\mathcal{I}} \int_{V_I^{\sigma_0}} \left|\nabla^2 v_{\sigma_0}\right|^2 \, dx \quad \le \quad c_{\mathbb{SH}} \mathcal{H}^1(\Gamma) \; + \; \varepsilon/2 \; . \qquad \text{(III.23)}$$

The final step will be to modify v_{σ_0} to get u_ε. The map v_{σ_0} is smooth on $\overline{\mathbb{B}^5} \setminus \operatorname{Sing} u$ and is degree-0 homogeneous about each point $a \in \operatorname{Sing} u$, in the ball $\mathbb{B}^5_{\delta_0/2}(a)$.

For each such a, consider the normalized map given by rescaling $v_{\sigma_0} \,|\, \partial\mathbb{B}^5_{\delta_0/2}(a)$, namely,

$$g_a \;:\; \mathbb{S}^4 \;\to\; \mathbb{S}^3 \;, \quad g_a(x) \;=\; v_{\sigma_0}[a + (\delta_0/2)x]$$

We claim that, in $\Pi_4(\mathbb{S}^3) \simeq \mathbb{Z}_2$, the homotopy class $[\![\, g_a]\!]$ is zero. To see this, suppose that $a = a_I$ and first note that the restriction of the original map $u \,|\, \partial\mathbb{B}^5_{\delta_0/2}(a)$ gives the nonzero class $[\![\mathbb{SH}]\!] \in \Pi_4(\mathbb{S}^3)$ by the definition of $\mathcal{R}$. Second, we slightly reparameterized $u \,|\, \partial\mathbb{B}^5_{\delta_0/2}(a)$ near the point $a + (\delta_0/2)\mathbf{e}_I$ to have constant value $\xi_{a_I} = (\mathbb{SH})(\mathbf{e}_I)$ in a small spherical cap of radius $\sigma_0\delta_0/2$. The resulting reparameterized map $\tilde{u}_a$ still induces the nonzero homotopy class in $\Pi_4(\mathbb{S}^3)$. Third, in forming the map $v_\sigma \,|\, \partial\mathbb{B}^5_{\delta_0/2}(a)$, we inserted a bubble in the small cap of constancy of $\tilde{u}_a$. This insertion gives the resulting sum in $\Pi_4(\mathbb{S}^3)$:

$$[\![g_a]\!] \;=\; [\![\tilde{u}_a]\!] + [\![\xi_{a_I} \star \mathbb{SH} \circ \Phi_{\rho_0}]\!] \;=\; [\![\mathbb{SH}]\!] + [\![\mathbb{SH} \circ \Phi_{\rho_0}]\!] \;=\; 2[\![\mathbb{SH}]\!] \;=\; 0$$

by (III.8). The same is true in case a is a second endpoint b_I.

Now, as in the proof of Lemma III.2, g_a is homotopic to a constant, and we may we may fix a smooth homotopy $h_a : [0,1] \times \mathbb{S}^4 \to \mathbb{S}^3$ so that

$$h_a(t,y) \;=\; \begin{cases} g_a(y) & \text{for } t \text{ near } 0 \\ (1,0,0,0) & \text{for } t \text{ near } 1 \\ . \end{cases}$$

Thus the map

$$H_a : \overline{\mathbb{B}^5} \to \mathbb{S}^3 \;, \qquad H_a(x) \;=\; h_a\,(1 - |x|,\; x/|x|) \;\text{ for } 0 < |x| \le 1 \;, \quad H_a(0) = (1,0,0,0) \;,$$

is smooth. Moreover, for $0 < \tau \le \delta_0/2$,

$$w_\tau \;:\; \bigcup_{a \in \operatorname{Sing} u} \mathbb{B}^5_\tau(a) \to \mathbb{S}^3 \;, \qquad w_\tau(x) = H_a\left(\frac{x-a}{\tau}\right) \text{ for } x \in \mathbb{B}^5_\tau(a) \;,$$

satisfies

$$\int_{\mathbb{B}^5_\tau(a)} |\nabla^2 w_\tau|^2 \, dx \;=\; \tau \int_{\mathbb{B}^5} |\nabla H_a|^2 \, dx \;,$$

and we can fix a positive $\tau_0 \le \delta_0/2$ so that

$$\sum_{a \in \operatorname{Sing} u} \int_{\mathbb{B}^5_{\tau_0}(a)} |\nabla^2 w_{\tau_0}|^2 \, dx \;<\; \varepsilon/2 \;. \tag{III.24}$$

Finally we define the desired map $u_\varepsilon : \mathbb{B}^5 \to \mathbb{S}^3$ by :

$$u_\varepsilon(x) \;=\; \begin{cases} v_{\sigma_0}(x) & \text{for } x \in \cup_{I \in \mathcal{I}} V_I^{\sigma_0} \setminus \cup_{a \in \operatorname{Sing} u} \mathbb{B}^5_{\tau_0}(a) \\ w_{\tau_0}(x) & \text{for } x \in \cup_{a \in \operatorname{Sing} u} \mathbb{B}^5_{\tau_0}(a) \\ u(x) & \text{otherwise .} \end{cases}$$

We easily verify that u_ε is smooth and coincides with u outside an ε neighborhood of Γ because $\sigma_0\delta_0/9 < \varepsilon$ and $\tau_0 \le \delta_0/2 < \varepsilon$. Moreover, by (III.23) and (III.24),

$$\begin{aligned} \int_{\mathbb{B}^5} |\nabla^2 u_\varepsilon|^2 \, dx \quad &\le \quad \sum_{I \in \mathcal{I}} \int_{V_I^{\sigma_0}} |\nabla^2 v_{\sigma_0}|^2 \, dx \;+\; \sum_{a \in \operatorname{Sing} u} \int_{\mathbb{B}^5_{\tau_0}(a)} |\nabla^2 w_{\tau_0}|^2 \, dx \;+\; \int_{\mathbb{B}^5} |\nabla^2 u|^2 \, dx \\ &\le \quad c_{\mathbb{SH}} \mathcal{H}^1(\Gamma) + \frac{\varepsilon}{2} + \frac{\varepsilon}{2} + \int_{\mathbb{B}^5} |\nabla^2 u|^2 \, dx \;. \end{aligned}$$

■

Corollary III.1 *If u and u_ε are as in Theorem III.1, then u_ε approaches u , $W^{2,2}$ weakly as $\varepsilon \to 0$.*

Proof. One has the strong L^2 convergence $\lim_{\varepsilon\to 0} \|u_\varepsilon - u\|_{L^2} = 0$, because the u_ε are uniformly bounded (by 1) and approach u pointwise on $\mathbb{B}^5 \setminus \operatorname{Sing} u$. Moreover, for any sequence $1 \geq \varepsilon_i \downarrow 0$, we have by Theorem III.1 and Lemma II.1, the bound

$$\sup_i \|u_{\varepsilon_i}\|^2_{W^{2,2}} \ < \ c_m \left(4 \ + \int_{\mathbb{B}^5} |\nabla^2 u|^2 \, dx + c_{\mathbb{SH}} \mathcal{H}^1(\Gamma)\right) \ < \ \infty \ .$$

By the weak*(=weak) compactness of the closed ball in $W^{2,2}(\mathbb{B}^5, \mathbb{R}^\ell)$, the sequence u_{ε_i} contains a subequence $u_{\varepsilon_{i'}}$ that is $W^{2,2}$ weakly convergent to some $w \in W^{2,2}(\mathbb{B}^5, \mathbb{R}^\ell)$. But, w, being by Rellich's theorem, the strong L^2 limit of the $u_{\varepsilon_{i'}}$, must necessarily be the original map u. Since any subsequence of u_ε subconverges to the same limit u and since the weak* (=weak) $W^{2,2}$ topology on bounded sets is metrizable, the original family u_ε converges $W^{2,2}$ weakly to u. ■

Remark III.1 *In Theorem III.1, one may replace $c_{\mathbb{SH}}$ by the optimal constant*

$$\tilde{c}_{\mathbb{SH}} \ = \ \inf \left\{ \int_{\mathbb{S}^4} \left|\nabla^2_{tan}\omega\right|^2 d\mathcal{H}^4 \ : \omega \in \mathcal{C}^\infty(\mathbb{S}^4, \mathbb{S}^3) \text{ and } [\![\omega]\!] = [\![\mathbb{SH}]\!] \right\} \ .$$

Here, for any ω as above, we can first $W^{2,2}$ strongly approximate u by a map which equals $\omega(x-a)/|x-a|$ in $\mathbb{B}_{\delta_1}(a)$ for all $a \in \operatorname{Sing} u$ and some $0 < \delta_1 << \delta_0$. Then we repeat the proofs with $\mathbb{SH}$ replaced by ω.

IV Connecting Singularities with Controlled Length

Suppose $u \in \mathcal{R}$ with $\operatorname{Sing} u = \{a_1, a_2, \ldots, a_m\}$ as above. Our goal in this section is to connect the singular points a_i, together in pairs or to $\partial\mathbb{B}^5$, by some union of curves whose total length is bounded by an absolute constant multiple of the *Hessian energy*, that is,

$$c \int_{\mathbb{B}^5} |\nabla^2 u|^2 \, dx \ .$$

This is therefore a bound on the length of a minimal connection for $\operatorname{Sing} u$, which will allow us, in Theorem V.3 below, to combine Lemma III.2 and Theorem III.1 to obtain the desired sequential weak density of $\mathcal{C}^\infty(\mathbb{B}^5, \mathbb{S}^3)$ in $W^{2,2}(\mathbb{B}^5, \mathbb{S}^3)$.

Using the surjectivity of the suspension of the Hopf map, we readily verify that each *regular* value $p \in \mathbb{S}^3 \setminus \{(-1,0,0,0), (1,0,0,0)\}$ of u gives a level surface

$$\Sigma = u^{-1}\{p\}$$

which necessarily contains all the singular points a_i of u. Note that $\Sigma = u^{-1}\{p\}$ is smoothly embedded away from the a_i with standard orientation $\omega_\Sigma \equiv *u^\# \omega_{\mathbb{S}^3}/|u^\#\omega_{\mathbb{S}^3}|$, induced from u. Concerning the behavior near a_i, the punctured neighborhood

$$\Sigma \cap \mathbb{B}_{\delta_0}(a_i) \setminus \{a_i\}$$

is simply a truncated cone whose boundary

$$\Gamma_i \ = \ \Sigma \cap \partial\mathbb{B}_{\delta_0}(a_i)$$

is a planar circle in the 3-sphere $\partial\mathbb{B}_{\delta_0}(a_i) \cap \left(\{\delta p_0\} \times \mathbb{R}^4\right)$ where $p = (p_0, p_1, p_2, p_3)$.

We will eventually choose the desired "connecting" curves all to lie on one such level surface Σ.

IV.1 Estimates for Choosing the Level Surface $\Sigma = u^{-1}\{p\}$

We first recall the 3 Jacobian $J_3 u = \| \wedge_3 Du\|$ and apply the coarea formula [Fe], §3.2.12 with

$$g \; = \; \frac{|\nabla u|^4 + |\nabla^2 u|^2}{J_3 u} \; ,$$

to obtain the relation

$$\int_{\mathbb{S}^3} \int_{u^{-1}\{p\}} \frac{|\nabla u|^4 + |\nabla^2 u|^2}{J_3 u} \, d\mathcal{H}^2 \, d\mathcal{H}^3 p \; = \; \int_{\mathbb{B}^5} \left(|\nabla u|^4 + |\nabla^2 u|^2 \right) dx \; . \qquad \text{(IV.25)}$$

Moreover, since $\|u\|_{L^\infty} = 1$, we also have (see [MR]) the integral inequality

$$\int_{\mathbb{B}^5} |\nabla u|^4 \leq \; c \int_{\mathbb{B}^5} |\nabla^2 u|^2 \, dx \; . \qquad \text{(IV.26)}$$

In case u is constant on $\partial \mathbb{B}^5$, we may verify this by computing

$$\begin{aligned}
\int_{\mathbb{B}^5} |\nabla u|^4 \; &= \; \int_{\mathbb{B}^5} (\nabla u \cdot \nabla u) \, |\nabla u|^2 \, dx \\
&= \; \int_{\mathbb{B}^5} \left[\operatorname{div} \left(u \nabla u |\nabla u|^2 \right) \; - \; u \cdot \Delta u |\nabla u|^2 \; - \; u \nabla u \cdot \nabla \left(|\nabla u|^2 \right) \right] dx \\
&\leq \; 0 \; + \; 5 \int_{\mathbb{B}^5} |\nabla^2 u| |\nabla u|^2 \, dx \; + \; 2 \int_{\mathbb{B}^5} |\nabla^2 u| |\nabla u|^2 \, dx \\
&\leq \; \frac{1}{2} \int_{\mathbb{B}^5} |\nabla u|^4 \, dx \; + \; \frac{49}{2} \int_{\mathbb{B}^5} |\nabla^2 u|^2 \, dx.
\end{aligned}$$

In the general case, we write $u = \sum_{i=1}^\infty \lambda_i u$ where $\{\lambda_i\}$ is a partition of unity adapted to a family of Whitney cubes for $\mathbb{B}^5$. See [MR]. (The above inequality is true even with the constraint $\|u\|_{BMO} \leq 1$ in place of $\|u\|_{L^\infty} \leq 1$ [MR].)

By (IV.25) and (IV.26) we may now choose a regular value $p \in \mathbb{S}^3$ of u so that

$$\int_{u^{-1}\{p\}} \frac{|\nabla u|^4 + |\nabla^2 u|^2}{J_3 u} \, d\mathcal{H}^2 \; \leq \; c \int_{\mathbb{B}^5} |\nabla^2 u|^2 \, dx \; . \qquad \text{(IV.27)}$$

By increasing c we will also insist that $|p_0|$ is small, say, $|p_0| \; < \; 1/100$. This smallness will be useful in guaranteeing that each tangent plane $\operatorname{Tan}(\Sigma, x)$, for $x \in \Sigma \cap \cup_{i=1}^m \mathbb{B}_{\delta_0}(a_i) \setminus \{a_i\}$, is close to $\{0\} \times \mathbb{R}^4$.

IV.2 A Pull-back Normal Framing for $\Sigma = u^{-1}\{p\}$

.

Suppose again that $p = (p_0, p_1, p_2, p_3) \in \mathbb{S}^3 \setminus \{(-1,0,0,0), \, (1,0,0,0)\}$ is a regular value of u. Then

$$\eta_1 \; = \; \left(-\sqrt{1 - p_0^2} \, , \, \frac{p_0 p_1}{\sqrt{1 - p_0^2}} \, , \, \frac{p_0 p_2}{\sqrt{1 - p_0^2}} \, , \, \frac{p_0 p_3}{\sqrt{1 - p_0^2}} \right)$$

is the unit vector tangent at p to the geodesic that runs from $(1,0,0,0)$ through p to $(-1,0,0,0)$. We may choose two other vectors

$$\eta_2 \, , \; \eta_3 \; \in \; \operatorname{Tan} \left(\{p_0\} \times \sqrt{1 - p_0^2} \, \mathbb{S}^2, p \right) \; \subset \; \operatorname{Tan}(\mathbb{S}^3, p)$$

so that η_1, η_2, η_3 becomes an orthonormal basis for $\mathrm{Tan}(\mathbb{S}^3, p)$. Since p is a regular value for u, these three vectors lift to three unique smooth linearly independent normal vectorfields τ_1, τ_2, τ_3 along $\Sigma = u^{-1}\{p\}$. That is, at each point $x \in \Sigma$,

$$\tau_j(x) \perp \Sigma \text{ at } x \quad \text{and} \quad Du(x)\,[\tau_j(x)] \;=\; \eta_j$$

for $j = 1, 2, 3$.

Near each singularity a_i the lifted vectorfields τ_1, τ_2, τ_3 are also orthonormal. In fact, for any point $x \in \Sigma \cap \mathbb{B}_{\delta_0}(a_i)$, $\frac{x_0 - a_{i0}}{|x - a_i|} = p_0$, and

$$\tau_1(x) \;=\; \left(-\sqrt{1-p_0^2}\,, \frac{p_0}{\sqrt{1-p_0^2}} \frac{x_1 - a_{i1}}{|x-a_i|}\,, \frac{p_0}{\sqrt{1-p_0^2}} \frac{x_2 - a_{i2}}{|x-a_i|}\,, \frac{p_0}{\sqrt{1-p_0^2}} \frac{x_3 - a_{i3}}{|x-a_i|} \right) . \tag{IV.28}$$

Also $\tau_1(x), \tau_2(x), \tau_3(x)$ are orthonormal for such x because the Hopf map is horizontally orthogonal and the lifts $\tau_2(x), \tau_3(x)$ are tangent to the 3 sphere $\{p_0\} \times \sqrt{1-p_0^2}\,\mathbb{S}^3$.

On the remainder of the surface $\Sigma \setminus \cup_{i=1}^m \mathbb{B}_{\delta_0}(a_i)$, the linearly independent vectorfields τ_1, τ_2, τ_3 are not necessarily orthonormal, and we use their Gram-Schmidt orthonormalizations

$$\begin{aligned}
\tilde\tau_1 &= \frac{\tau_1}{|\tau_1|}\ ,\\
\tilde\tau_2 &= \frac{\tau_2 - (\tilde\tau_1\cdot\tau_2)\tilde\tau_1}{|\tau_2 - (\tilde\tau_1\cdot\tau_2)\tilde\tau_1|} = \frac{\tau_2 - (\tilde\tau_1\cdot\tau_2)\tilde\tau_1}{|\tilde\tau_1\wedge\tau_2|}\ ,\\
\tilde\tau_3 &= \frac{\tau_3 - (\tilde\tau_1\cdot\tau_3)\tilde\tau_1 - (\tilde\tau_2\cdot\tau_3)\tilde\tau_2}{|\tau_3 - (\tilde\tau_1\cdot\tau_3)\tilde\tau_1 - (\tilde\tau_2\cdot\tau_3)\tilde\tau_2|} = \frac{\tau_3 - (\tilde\tau_1\cdot\tau_3)\tilde\tau_1 - (\tilde\tau_2\cdot\tau_3)\tilde\tau_2}{|\tilde\tau_1\wedge\tilde\tau_2\wedge\tau_3|}\ ,
\end{aligned}$$

which provide an *orthonormal framing* for the unit normal bundle of Σ.

We need to estimate the total variation of these orthonormalizations. Noting that $|\nabla\left(\frac{\tau}{|\tau|}\right)| \le 2\frac{|\nabla\tau|}{|\tau|}$ for any differentiable τ, we see that

$$\begin{aligned}
|\nabla\tilde\tau_1| &\le 2\frac{|\nabla\tau_1|}{|\tau_1|} \le 2\frac{|\nabla\tau_1||\tau_1||\tau_2||\tau_3|}{|\tau_1||\tau_1\wedge\tau_2\wedge\tau_3|} = 2\frac{|\tau_2||\tau_3||\nabla\tau_1|}{|\tau_1\wedge\tau_2\wedge\tau_3|}\ ,\\
|\nabla\tilde\tau_2| &= 2\left[\frac{\tau_2 - (\tilde\tau_1\cdot\tau_2)\tilde\tau_1}{|\tilde\tau_1\wedge\tau_2|}\right] \le 2\left[\frac{2|\nabla\tau_2| + 2|\tau_2||\nabla\tilde\tau_1|}{|\tau_1\wedge\tau_2||\tau_1|^{-1}}\right]\\
&\le 8\left[\frac{|\tau_1||\nabla\tau_2| + |\tau_2||\nabla\tau_1|}{|\tau_1\wedge\tau_2|}\cdot\frac{|\tau_1\wedge\tau_2||\tau_3|}{|\tau_1\wedge\tau_2\wedge\tau_3|}\right] = 8\left[\frac{|\tau_2||\tau_3||\nabla\tau_1| + |\tau_1||\tau_3||\nabla\tau_2|}{|\tau_1\wedge\tau_2\wedge\tau_3|}\right]\ ,\\
|\nabla\tilde\tau_3| &\le 2\left[\frac{3|\nabla\tau_3| + 2|\tau_3||\nabla\tilde\tau_1| + 2|\tau_3||\nabla\tilde\tau_2|}{|\tilde\tau_1\wedge\tilde\tau_2\wedge\tau_3|}\right]\\
&\le 32\left[\frac{|\nabla\tau_3| + |\tau_3||\tau_1|^{-1}|\nabla\tau_1| + |\tau_3|\left(\frac{|\tau_1||\nabla\tau_2| + |\tau_2||\nabla\tau_1|}{|\tau_1\wedge\tau_2|}\right)}{|\frac{\tau_1}{|\tau_1|}\wedge\left(\frac{\tau_2}{||\tau_1|^{-1}\tau_1\wedge\tau_2|}\right)\wedge\tau_3|}\right]\\
&\le 32\left[\frac{|\tau_1||\tau_2||\nabla\tau_3| + |\tau_2||\tau_3||\nabla\tau_1| + |\tau_1||\tau_3||\nabla\tau_2|}{|\tau_1\wedge\tau_2\wedge\tau_3|}\right]\ .
\end{aligned}$$

Inasmuch as

$$|\tau_j| \le |\nabla u|\ , \quad |\nabla\tau_j| \le |\nabla^2 u|\ , \quad |\tau_1\wedge\tau_2\wedge\tau_3| = J_3 u\ ,$$

we deduce the general pointwise estimate

$$|\nabla\tilde\tau_j| \ \le\ c\frac{|\nabla u|^2|\nabla^2 u|}{|J_3 u|} \ \le\ c\frac{|\nabla u|^4+|\nabla^2 u|^2}{|J_3 u|}\ ,$$

which we may integrate using (IV.27) to obtain the variation estimate along $\Sigma = u^{-1}\{p\}$,

$$\int_\Sigma |\nabla\tilde\tau_j|\, d\mathcal{H}^2 \ \le\ c\int_{\mathbb{B}^5}|\nabla^2 u|^2\, dx\ . \tag{IV.29}$$

IV.3 Twisting of the Normal Frame $\tilde\tau_1, \tilde\tau_2, \tilde\tau_3$ About Each Singularity a_i

.

First we recall from[MS], §5-6 that the Grassmannian

$$\tilde G_2(\mathbb{R}^5)$$

of *oriented* 2 planes through the origin in $\mathbb{R}^5$ is a compact smooth manifold of dimension 6. It may be identified with the set of simple unit 2 vectors in $\mathbb{R}^5$,

$$\{v\wedge w\in\wedge_2\mathbb{R}^5\ :\ v\in\mathbb{S}^4,\ w\in\mathbb{S}^4,\ v\cdot w = 0\}\ .$$

We will use the distance $|P-Q|$ on $\tilde G_2(\mathbb{R}^5)$ given by this embedding into $\wedge_2\mathbb{R}^5\approx\mathbb{R}^{10}$.

For a fixed plane $P\in\tilde G_2(\mathbb{R}^5)$, the set of *nontransverse* 2 planes

$$\mathcal{Q}_P\ =\ \Big\{Q\in\tilde G_2(\mathbb{R}^5)\ :\ P\cap Q\ne\{0\}\Big\}$$

is a (Schubert) subvariety of dimension $1+3=4$ because every $Q\in\mathcal{Q}_P\setminus\{P\}$ equals $v\wedge w$ for some $w\in\mathbb{S}^4\cap P$ and some $v\in\mathbb{S}^4\cap w^\perp$. These subvarieties are all orthogonally isomorphic and, in particular, have the same finite 4 dimensional Hausdorff measure. Also

$$Y_P\ =\ \{Q\in\mathcal{Q}_P\ :\ P^\perp\cap Q\ne\{0\}\}$$

is a closed subvariety of dimension 3, and $\mathcal{Q}_P\setminus Y_P$ is a smooth submanifold.

Then, near each singularity a_i, the set of 2 planes nontransverse to the cone $\Sigma\cap\mathbb{B}_{\delta_0}(a_i)\setminus\{a_i\}$,

$$W\ =\ \bigcup_{x\in\Sigma\cap\mathbb{B}_{\delta_0}(a_i)\setminus\{a_i\}}\mathcal{Q}_{\mathrm{Tan}(\Sigma,x)}\ =\ \bigcup_{x\in\Gamma_i}\mathcal{Q}_{\mathrm{Tan}(\Sigma,x)}\ ,$$

has dimension only $1+4=5<6=\dim\tilde G_2(\mathbb{R}^5)$. Note also its location, that W is, by the smallness of $|p_0|$, contained in the tubular neighborhood

$$V\ \equiv\ \{Q\in\tilde G_2(\mathbb{R}^5)\ :\ \mathrm{dist}\Big(Q,\tilde G_2(\{0\}\times\mathbb{R}^4\Big)<1/50\},$$

of the 4 dimensional subgrassmannian $\tilde G_2(\{0\}\times\mathbb{R}^4)$.

We now describe explicitly how the framing $\tilde\tau_1(x),\tilde\tau_2(x),\tilde\tau_3(x)$ *twists once* as x goes around each circle Γ_i. The problem is that the vectors $\tilde\tau_j(x)$ lie in the normal space $\mathrm{Nor}(\Sigma,x)$ which also varies with x. To measure the rotation of the frame $\tilde\tau_1(x),\tilde\tau_2(x),\tilde\tau_3(x)$, as x traverses the circle Γ_i, it is necessary to use some *reference frame* for $\mathrm{Nor}(\Sigma,x)$.

We can induce such a frame from some fixed unit vectors in $\mathbb{R}^5$ as follows: Consider a fixed $Q \in \tilde{G}_2(\mathbb{R}^5) \setminus W$, and suppose $Q = v \wedge w$ with v, w being an orthonormal basis for Q. For each $x \in \Gamma_i$, the orthogonal projections of v, w onto $\mathrm{Nor}(\Sigma, x)$ are linearly independent; let $\sigma_1(x)$, $\sigma_2(x)$ be their Gram-Schmidt orthonormalizations. We then get $\sigma_3(x)$ by using the map u to pull-back the orientation of $\mathbb{S}^3$ to $\mathrm{Nor}(\Sigma, x)$ so that the resulting orienting 3 vector is $\sigma_1(x) \wedge \sigma_2(x) \wedge \sigma_3(x)$ for a unique unit vector $\sigma_3(x) \in \mathrm{Nor}(\Sigma, x)$ orthogonal to $\sigma_1(x)$, $\sigma_2(x)$. We view

$$\sigma_1(x)\,,\ \sigma_2(x)\,,\ \sigma_3(x)$$

as the *reference frame* determined by the fixed vectors v, w. For each $x \in \Gamma_i$, there is then a unique rotation $\gamma(x) \in \mathbb{SO}(3)$ so that

$$\gamma(x)\,[\sigma_j(x)] \;=\; \tilde{\tau}_j(x) \quad \text{for} \quad j = 1, 2, 3\ .$$

In the next paragraph we will check that $\gamma : \Gamma_i \to \mathbb{SO}(3)$ is a single geodesic circle in $\mathbb{SO}(3)$. The twisting of the frame $\tilde{\tau}_1, \tilde{\tau}_2, \tilde{\tau}_3$ around the circle Γ_i is reflected in the fact that such a circle induces the nonzero element in $\Pi_1(\mathbb{SO}(3)) \simeq \mathbb{Z}_2$.

In the special case $v = (1, 0, 0, 0)$, the normalized orthogonal projection of v onto $\mathrm{Nor}(\Sigma, x)$ is, by (IV.4), simply

$$\sigma_1(x) \;=\; \tilde{\tau}_1(x)\ .$$

So in this case, each orthogonal matrix $\gamma(x)$ is a rotation about the first axis, and one checks that, as x traverses the circle Γ_i once, these rotations complete a single geodesic circle in $\mathbb{SO}(3)$. For another choice of v, the geodesic circle $\gamma : \Gamma_i \to \mathbb{SO}(3)$ involves a circle of rotations about a different axis combined with a single orthogonal change of coordinates.

IV.4 A Reference Normal Framing for $\Sigma = u^{-1}\{p\}$

.

The above calculations near the a_i suggest comparing on the *whole* surface $u^{-1}\{p\}$ the pull-back normal framing $\tilde{\tau}_1(x), \tilde{\tau}_2(x), \tilde{\tau}_3(x)$ with some reference normal framing $\sigma_1(x), \sigma_2(x), \sigma_3(x)$ induced by two fixed vectors v, w. Unfortunately, there may not exist fixed vectors v, w so that the corresponding reference framing σ_1, σ_2, σ_3 is defined *everywhere* on Σ. In this section we show that any orthonormal basis v, w of almost every oriented 2 plane $Q \in \tilde{G}_2(\mathbb{R}^5)$ gives a reference framing on Σ which is well-defined and smooth except at finitely many *discontinuities*

$$b_1, b_2, \ldots, b_n\ .$$

We will then need to connect the original singularities a_i to the b_j (or to $\partial\mathbb{B}^5$) and, in §IV.6, choose other curves to connect the b_j to each other (or to $\partial\mathbb{B}^5$), with all curves having total length bounded by a multiple of $\int_{\mathbb{B}^5} |\nabla^2 u|^2\, dx$.

To find a suitable $Q = v \wedge w$, we will first rule out the exceptional planes that contain some nonzero vector normal to Σ at some point $x \in \Sigma$. The really exceptional 2 planes that lie completely in some normal space

$$X \;=\; \cup_{x\in\Sigma} X_x \quad \text{where} \quad X_x \;=\; \{Q \in \tilde{G}_2(\mathbb{R}^5)\ :\ Q \subset \mathrm{Nor}(\Sigma, x)\}\ .$$

Then X has dimension at most $2 + 2 = 4 < 6 = \dim \tilde{G}_2(\mathbb{R}^5)$ because $\dim \Sigma = 2$ and $\dim \tilde{G}_2(\mathbb{R}^3) = 2$. The remaining set of exceptional planes

$$Y \;=\; \cup_{x\in\Sigma} Y_x \quad \text{where} \quad Y_x \;=\; \{Q \in \tilde{G}_2(\mathbb{R}^5)\ :\ \dim\,(Q \cap \mathrm{Nor}(\Sigma, x)) = 1\}$$

has dimension at most $2 + 2 + 1 = 5 < 6 = \dim \tilde{G}_2(\mathbb{R}^5)$ because

$$Y_x = \ \{e \wedge w\ :\ e \in \mathbb{S}^4 \cap \mathrm{Nor}(\Sigma, x) \text{ and } \ w \in \mathbb{S}^4 \cap \mathrm{Tan}(\Sigma, x)\}\ .$$

In terms of our previous notation, $Y_{\mathrm{Tan}(\Sigma,x)} = X_x \cup Y_x$.

Any unit vector $e \notin \mathrm{Nor}(\Sigma, x)$ has a nonzero orthogonal projection

$$e_T(x)$$

onto $\mathrm{Tan}(\Sigma, x)$.

Normalizing

$$\tilde{e}_T(x) \ = \ \frac{e_T(x)}{|e_T(x)|} \ ,$$

we find a unique unit vector $e_\Sigma(x) \in \mathrm{Tan}(\Sigma, x)$ orthogonal to $e_T(x)$ so that $\tilde{e}_T(x) \wedge e_\Sigma(x)$ is the standard orientation of $\mathrm{Tan}(\Sigma, x)$. Then

$$e \cdot e_\Sigma(x) \ = \ (e - e_T(x)) \cdot e_\Sigma(x) \ + \ e_T(x) \cdot e_\Sigma(x) \ = \ 0 \ + \ 0$$

because $e - e_T(x) \in \mathrm{Nor}(\Sigma, x)$. Thus,

$$\tilde{e}_T(x), \ e_\Sigma(x), \tilde{\tau}_1(x), \ \tilde{\tau}_2(x), \ \tilde{\tau}_3(x),$$

is an orthonormal basis for $\mathbb{R}^5$.

Away from the 4 dimensional unit normal bundle

$$\mathcal{N}_\Sigma \ = \ \{(x, e) \ : \ x \in \Sigma, \ e \in \mathbb{S}^4 \cap \mathrm{Nor}(\Sigma, x)\} \ ,$$

we now define the basic map

$$\Phi \ : \ (\Sigma \times \mathbb{S}^4) \setminus \mathcal{N}_\Sigma \ \to \ G_2(\mathbb{R}^5) \ , \quad \Phi(x, e) = e \wedge e_\Sigma(x) \ ,$$

to parameterize the planes *nontransverse to* Σ in $\tilde{G}_2(\mathbb{R}^5) \setminus Y$. Incidentally, these do include the 2 dimensional family of tangent planes

$$Z \ = \ \{Q \in \tilde{G}_2(\mathbb{R}^5) \ : \ Q = \mathrm{Tan}(\Sigma, x) \text{ for some } x \in \Sigma\} \ .$$

In terms of the notation at the beginning of this section, for any 2 plane $Q \notin Y$,

$$Q \in \mathcal{Q}_{\mathrm{Tan}(\Sigma,x)} \iff Q \ = \ \Phi(x, e) \text{ for some } \ e \in \mathbb{S}^4 \setminus \mathrm{Nor}(\Sigma, x) \ .$$

Note that $\Phi(x, -e) = \Phi(x, e)$, and, in fact,

$$\Phi(x, e') = \Phi(x, e) \in \tilde{G}_2(\mathbb{R}^5) \setminus Y \quad \iff \quad e' = \pm e \ .$$

It is also easy to describe the behavior of Φ at the singular set $\mathcal{N}_\Sigma$. A 2 plane Q belongs to Y, that is, $Q = v \wedge w$ for some $v \in \mathrm{Nor}(\Sigma, x) \cap \mathbb{S}^4$ and $w \in \mathrm{Tan}(\Sigma, x) \cap \mathbb{S}^4$, if and only if $Q = \lim_{n\to\infty} \Phi(x_n, v_n)$ for some sequence $(x_n, v_n) \in (\Sigma \times \mathbb{S}^4) \setminus \mathcal{N}_\Sigma$ approaching (x, v). The map Φ essentially "blows-up" the 4 dimensional $\mathcal{N}_\Sigma$ to the 5 dimensional Y, and, in particular, *any smooth curve in* $\tilde{G}_2(\mathbb{R}^5)$ *transverse to* Y *lifts by* Φ *to a pair of antipodal curves in* $\Sigma \times \mathbb{S}^4$ *extending continuously transversally across* $\mathcal{N}_\Sigma$.

We now choose and fix $Q \in G^2(\mathbb{R}^5)$ so that *neither* Q *nor* $-Q$ *belong to the* 5 *dimensional exceptional set* $X \cup Y \cup Z$ *and both are regular values of* Φ. We may also insist that Q is close to the 3 dimensional Schubert cycle

$$H \ = \ \{(1, 0, \ldots, 0) \wedge (0, v_1, v_2, v_3, v_4) \ : \ v_1^2 + \cdots + v_4^2 = 1\} \ ,$$

say $\mathrm{dist}\,(Q, H) \ < \ 1/100$. This will guarantee that Q is well separated from the open region V that contains W.

Since

$$\dim(\Sigma \times \mathbb{S}^4) = 6 = \dim \tilde{G}_2(\mathbb{R}^5) ,$$

$\Phi^{-1}\{Q, -Q\}$ is a finite set, say

$$\Phi^{-1}\{Q,-Q\} \;=\; \{(b_1, e_1), (b_1, -e_1), (b_2, e_2), (b_2, -e_2), \ldots, (b_n, e_n), (b_n, -e_n)\} .$$

We now see that the reference framing $\sigma_1(x), \sigma_2(x), \sigma_3(x)$ of $\mathrm{Nor}(\Sigma, x)$ corresponding to any fixed orthonormal basis v, w of Q fails to exist precisely at the points $b_1, b_2, \ldots, b_n$. As before, we now have the smooth mapping

$$\gamma \;:\; \Sigma \setminus \{a_1, \ldots, a_m, b_1, \ldots, b_n\} \;\to\; \mathbb{SO}(3) ,$$

which is defined by the condition $\gamma(x)\,[\sigma_j(x)] \;=\; \tilde{\tau}_j(x)$ for $j = 1, 2, 3$ or, in column-vector notation,

$$\gamma \;=\; [\sigma_1\sigma_2\sigma_3]^{-1}\,[\tilde{\tau}_1\tilde{\tau}_2\tilde{\tau}_3] \;.$$

IV.5 Asymptotic Behavior of γ Near the Singularities a_i and b_j

.

As discussed in §3.1, the map u, the surface $\Sigma = u^{-1}\{p\}$, the frames $\tilde{\tau}_1, \tilde{\tau}_2, \tilde{\tau}_3$ and $\sigma_1, \sigma_2, \sigma_3$, and the rotation field γ are all precisely known near a singularity a_i in the cone neighborhood $\Sigma \cap \mathbb{B}_{\delta_0}(a_i) \setminus \{a_i\}$. In particular, γ is homogeneous of degree 0 on $\Sigma \cap \mathbb{B}_{\delta_0}(a_i) \setminus \{a_i\}$; on its boundary $\gamma|\,\Gamma_i$ is a constant-speed geodesic circle.

At each b_j, the frame $\tilde{\tau}_1, \tilde{\tau}_2, \tilde{\tau}_3$ is smooth, but the frame $\sigma_1, \sigma_2, \sigma_3$, and hence the rotation γ, has an essential discontinuity. Nevertheless, we may deduce some of the asymptotic behavior at b_j because $\pm Q$ were chosen to be regular values of Φ. In fact, we'll verify:
The tangent map γ_j of γ at b_j,

$$\gamma_j : Tan(\Sigma, b_j) \cap \mathbb{B}_1(0) \to \mathbb{SO}(3) , \quad \gamma_j(x) = \lim_{r \to 0} \gamma \left[\exp^{\Sigma}_{b_j}(rx) \right] ,$$

exists and is the homogeneous degree 0 extension of some reparameterization of a geodesic circle in $\mathbb{SO}(3)$. In particular, for small positive δ, $\gamma\,|\,(\Sigma \cap \partial\mathbb{B}_\delta(b_j))$ is an embedded circle inducing the nonzero element of $\Pi_1(\mathbb{SO}(3)) \simeq \mathbb{Z}_2$.

To check this, we use, as above, the more convenient orthonormal basis $\{e_j, e_{j\Sigma}\}$ for Q; that is,

$$e_{j\Sigma} = e_{j\Sigma}(b_j) \in \mathrm{Tan}(\Sigma, b_j) \;\;\text{and}\;\; Q = e_j \wedge e_{j\Sigma} = \Phi(b_j, \pm e_j) .$$

Then, for $x \in \Sigma$, let

$$e_j^N(x) , \quad e_{j\Sigma}^N(x)$$

denote the orthogonal projections of the fixed vectors e_j, $e_{j\Sigma}$ onto $\mathrm{Nor}(\Sigma, x)$, and

$$\hat{e}_j^N(x)$$

denote the cross-product of $e_{j\Sigma}^N(x)$ and $e_j^N(x)$ in $\mathrm{Nor}(\Sigma, x)$. These three vectorfields are smooth near b_j with

$$e_j^N(b_j) \;\neq\; 0 , \quad e_{j\Sigma}^N(b_j) \;=\; 0 , \quad \hat{e}_j^N(b_j) \;=\; 0 .$$

Here our insistence that $\pm Q \notin Z$ guarantees that Q is not tangent to Σ at b_j. Let g_j denote the orthogonal projection of $\mathbb{R}^5$ onto the 2 plane

$$P_j \;=\; \mathrm{Nor}(\Sigma, b_j) \cap \left[e_j^N(b_j)\right]^{\perp} .$$

Then $G_j(x) = g_j \circ e_j^N(x)$ defines a smooth map from a Σ neighborhood of b_j to P_j, which has, by the regularity of Φ at $(b_j, \tilde{e}_j)$, a simple, nondegenerate zero at b_j (of degree ± 1). It follows that as x circulates $\Sigma \cap \partial\mathbb{B}_\delta(b_j)$ once, for δ small, $G_j(x)$ and similarly $g_j \circ \hat{e}_j^N(x)$, circulate 0 once in P_j. Returning to the original basis v, w of Q, we now check that, as x circulates $\Sigma \cap \partial\mathbb{B}_\delta(b_j)$ once, the frame $\sigma_1(x), \sigma_2(x), \sigma_3(x)$ approximately, and asymptotically as $\delta \to 0$, rotates once about the vector $e_j^N(b_j)$. Since the frame $\tilde{\tau}_1(x), \tilde{\tau}_2(x), \tilde{\tau}_3(x)$ is smooth at b_j, we see that the map γ has, at b_j, a tangent map γ_j as described above.

IV.6 Connecting the Singularities a_i to the b_j or to $\partial\mathbb{B}^5$

.

Here we will find curves reaching all the a_i and b_j. Concerning the a_i, we recall from [Br],§III,10] that $\mathbb{SO}(3)$ is isometric to $\mathbb{RP}^3 \simeq \mathbb{S}^3/\{x \sim -x\}$. Any geodesic circle Γ in $\mathbb{SO}(3)$ generates $\Pi_1(\mathbb{SO}(3)) \simeq \mathbb{Z}_2$ and lifts to a great circle $\tilde{\Gamma}$ in $\mathbb{S}^3$. The rotations at maximal distance from Γ form another geodesic circle $\Gamma^\perp$ and the nearest point retraction

$$\rho_\Gamma : \mathbb{SO}(3) \setminus \Gamma \ \to \ \Gamma^\perp$$

is induced by the standard nearest point retraction

$$\rho_{\tilde{\Gamma}} : \mathbb{S}^3 \setminus \tilde{\Gamma} \ \to \ \tilde{\Gamma}^\perp \ .$$

In particular,

$$|\nabla \rho_\Gamma(\zeta)| \ \le \ \frac{c}{\mathrm{dist}(\zeta, \Gamma)} \quad \text{for} \quad \zeta \in \mathbb{SO}(3) \ . \tag{IV.30}$$

Any geodesic circle Γ' in $\mathbb{SO}(3)$ that does *not* intersect Γ is mapped diffeomorphically by ρ_Γ onto the circle $\Gamma^\perp$. We deduce that if Γ is chosen to miss the asymptotic circles

$$\gamma(\Gamma_i) \quad \text{and} \quad \gamma_j\left(\mathrm{Tan}(\Sigma, b_j) \cap \mathbb{S}^4\right)$$

associated with the singularities a_i and b_j, then, on Σ, the composition $\rho_\Gamma \circ \gamma$ maps every sufficiently small circle

$$\Sigma \cap \partial\mathbb{B}_\delta(a_i) \quad \text{and} \quad \Sigma \cap \partial\mathbb{B}_\delta(b_j)$$

diffeomorphically onto the circle $\Gamma^\perp$.

Under the identification of $\mathbb{SO}(3)$ with $\mathbb{RP}^3$, $\mathbb{SO}(4)$ acts transitively by isometry on

$$\mathcal{G} \ = \ \{\text{geodesic circles } \Gamma \subset \mathbb{SO}(3) \ \} \ .$$

Then $\mathcal{G}$ is compact and admits a positive invariant measure $\mu_\mathcal{G}$. For $\mu_\mathcal{G}$ almost every circle Γ,

$$\Gamma \cap \gamma(\Gamma_i) = \emptyset \ \text{ for } \ i = 1, \ldots, m \ , \qquad \Gamma \cap \gamma_j\left(\mathrm{Tan}(\Sigma, b_j) \cap \mathbb{S}^4\right) = \emptyset \ \text{ for } \ j = 1, \ldots, n \ ,$$

and Γ is transverse to the map γ. In particular, $\gamma^{-1}(\Gamma)$ is a finite subset

$$\{c_1, c_2, \ldots, c_\ell\}$$

of Σ. For such a circle Γ and any regular value $z \in \Gamma^\perp$ of

$$\rho_\Gamma \circ \gamma \ : \ \Sigma \setminus \{a_1, \ldots, a_m, b_1, \ldots, b_n, c_1, \ldots, c_\ell\} \ \to \ \Gamma^\perp \ ,$$

the fiber

$$A \ = \ (\rho_\Gamma \circ \gamma)^{-1}\{z\}$$

is a smooth embedded 1 dimensional submanifold with

$$(\mathrm{Clos}\,A)\setminus A \;\subset\; \{a_1,\dots,a_m,b_1,\dots,b_n,c_1,\dots,c_\ell\}\cup\partial\mathbb{B}^5 \ .$$

We also can deduce the local behavior of A near each of the points a_i, b_j, c_k. From the above description of the asymptotic behavior of γ near a_i and b_j, we see that

$$\mathbb{B}_{\delta_0}(a_i)\cap \mathrm{Clos}\,A$$

is simply a *single line segment with one* __*endpoint*__ a_i while

$$\mathbb{B}_{\delta}(b_j)\cap \mathrm{Clos}\,A$$

is, for δ sufficiently small, a *single smooth segment with one* __*endpoint*__ b_j. On the other hand,

$$\mathbb{B}_{\delta}(c_k)\cap \mathrm{Clos}\,A$$

is, for δ sufficiently small, a *single smooth segment with an* __*interior*__ *point* c_k. To see this, observe that, for the lifted map $\rho_{\tilde\Gamma}:\mathbb{S}^3\setminus\tilde\Gamma\to\tilde\Gamma^\perp$ and any point $\tilde z\in\tilde\Gamma^\perp$, the fiber $\rho_{\tilde S}^{-1}\{z\}$ is an open great hemisphere, centered at z, with boundary $\tilde\Gamma$. It follows for the downstairs map ρ_Γ that $E_z = \mathrm{Clos}\left(\rho_\Gamma^{-1}\{z\}\right)\}$ is a full geodesic 2 sphere containing z and the circle Γ. Since the surface $\gamma(\Sigma)$ intersects the circle Γ transversely at a finite set, this sphere E_z is also transverse to $\gamma(\Sigma)$ near this set. Thus, for δ sufficiently small, $\mathbb{B}_\delta(c_k)\cap\mathrm{Clos}\,A$, being mapped diffeomorphically by γ onto the intersection $E_z\cap\gamma\left(\Sigma\cap\mathbb{B}_\delta(c_k)\right)$, is an open smooth segment containing c_k in its interior.

Combining this boundary behavior with the interior smoothness of the 1 manifold A, we now conclude that

$\mathbb{B}^5\cap$ *Clos* A *globally consists of disjoint smooth segments joining pairs of points from*

$$\{a_1,\dots,a_m,b_1,\dots,b_n\}\cup\partial\mathbb{B}^5 \ .$$

Moreover, each point a_i *or* b_j *is the endpoint of precisely one segment.*

IV.7 Estimating the Length of the Connecting Set A

.

The definition of the A depends on many choices:

(1) the *point* $p\in\mathbb{S}^3$ near $\{0\}\times\mathbb{R}^4$, which determines the surface $\Sigma = u^{-1}\{p\}$,

(2) the *vectors* $\eta_2,\eta_2,\eta_3\in\mathrm{Tan}(\mathbb{S}^3,p)$, which determine the pull-back normal framing $\tilde\tau_1,\tilde\tau_2,\tilde\tau_3$,

(3) the *vectors* v, $w\in\mathbb{S}^4$, which determine the reference normal framing $\sigma_1,\sigma_2,\sigma_3$ and the rotation field $\gamma = [\sigma_1\sigma_2\sigma_3]^{-1}\,[\tilde\tau_1\tilde\tau_2\tilde\tau_3] \ : \ \Sigma\setminus\{b_1,\dots,b_m\}\to\mathbb{SO}(3)$,

(4) the *circle* $\Gamma\subset\mathbb{SO}(3)$, which determines the retraction $\rho_\Gamma:\mathbb{SO}(3)\setminus\Gamma\to\Gamma^\perp$, and

(5) the *point* $z\in\Gamma^\perp$, which finally gives $A=(\rho_\Gamma\circ\gamma)^{-1}\{z\}$.

We need to make suitable choices of these to get the desired length estimate for A. In §IV.1 we already used one coarea formula to choose $p\in\mathbb{S}^3$ to give the basic estimate (IV.25)

$$\int_\Sigma \frac{|\nabla u|^4+|\nabla^2 u|^2}{J_3 u}\,d\mathcal{H}^2 \ \le\ c\int_{\mathbb{B}^5}|\nabla^2 u|^2\,dx \ ,$$

and the pull-back frame estimate (IV.29)

$$\int_\Sigma |\nabla\tilde{\tau}_j|\, d\mathcal{H}^2 \;\le\; c\int_{\mathbb{B}^5} |\nabla^2 u|^2\, dx \;,$$

independent of the choice of η_1, η_2, η_3, then followed. For the choice of $z \in S^\perp$, we want to use another coarea formula, ([Fe], §3.2.22)

$$\int_{\Gamma^\perp} \mathcal{H}^1(\rho_\Gamma \circ \gamma)^{-1}\{z\}\, dz \;=\; \int_\Sigma |\nabla(\rho_\Gamma \circ \gamma)|\, d\mathcal{H}^2 \;. \tag{IV.31}$$

To bound the righthand integral, we first use the chain rule and (IV.30) for the pointwise estimate

$$|\nabla(\rho_\Gamma \circ \gamma)(x)| \;=\; |\nabla(\rho_\Gamma)\,(\gamma(x))\,||\nabla\gamma(x)| \;\le\; \frac{c}{\mathrm{dist}\,(\gamma(x),\Gamma)}|\nabla\gamma(x)| \;. \tag{IV.32}$$

Next we observe the finiteness of the integral

$$C \;=\; \int_{\mathcal{G}} \frac{1}{\mathrm{dist}(\zeta,\Gamma)}\, d\mu_{\mathcal{G}}\Gamma \;<\; \infty \;,$$

independent of the point $\zeta \in \mathbb{SO}(3)$. To verify this, we note that $\mu_{\mathcal{G}}(\mathcal{G}) < \infty$ and choose a smooth coordinate chart for $\mathbb{SO}(3)$ near ζ that maps ζ to $0 \in \mathbb{R}^3$ and that transforms circles into affine lines in $\mathbb{R}^3$. Distances are comparable, and an affine line in $\mathbb{R}^3 \setminus \{0\}$ is described by its nearest point a to the origin and a direction in the plane $a^\perp$. Since

$$\mu_{\mathcal{G}}\{\Gamma \in \mathcal{G} \;:\; \zeta \in \Gamma\} \;=\; 0 \;,$$

the finiteness of C now follows from the finiteness of the 3 dimensional integral

$$\int_{\mathbb{R}^3 \cap \mathbb{B}_1} |y|^{-1}\, dy \;.$$

We deduce from Fubini's Theorem, (IV.31), and (IV.32) that

$$\begin{aligned}\int_{\mathcal{G}}\int_{\Gamma^\perp} \mathcal{H}^1(\rho_\Gamma \circ \gamma)^{-1}\{z\}\, dz\, d\mu_{\mathcal{G}}\Gamma \quad &\le\quad c\int_\Sigma |\nabla\gamma(x)| \int_{\mathcal{G}} \frac{1}{\mathrm{dist}\,(\gamma(x),\Gamma)}\, d\mu_{\mathcal{G}}\Gamma\, d\mathcal{H}^2 x \\ &\le\quad c\,C\int_\Sigma |\nabla\gamma(x)|\, d\mathcal{H}^2 x \;.\end{aligned}$$

Thus there exists a $\Gamma \in \mathcal{G}$ and $z \in \Gamma^\perp$ so that

$$\mathcal{H}^1(\rho_\Gamma \circ \gamma)^{-1}\{z\} \;\le\; c\int_\Sigma |\nabla\gamma(x)|\, d\mathcal{H}^2 x \;. \tag{IV.33}$$

To estimate the righthand side, recall the matrix formula

$$\gamma \;=\; [\sigma_1\sigma_2\sigma_3]^{-1}\,[\tilde{\tau}_1\tilde{\tau}_2\tilde{\tau}_3] \;.$$

and use Cramer's rule and the product and quotient rules to deduce the pointwise bound

$$|\nabla\gamma(x)| \;\le\; c\sum_{j=1}^{3}\left(\,|\nabla\sigma_j(x)| \;+\; |\nabla\tilde{\tau}_j(x)|\,\right) \;. \tag{IV.34}$$

In light of (IV.29), it remains to bound each term $\int_\Sigma |\nabla\sigma_j(x)|\, d\mathcal{H}^2 x$ for $j = 1, 2, 3$.

For the first one, note that

$$|\nabla\sigma_1| \;=\; |\nabla\left(\frac{v^N}{|v^N|}\right)| \;\le\; 2\frac{|\nabla v^N|}{|v^N|} \tag{IV.35}$$

where $v^N(x)$ is the orthogonal projection of v onto the normal space $\mathrm{Nor}(\Sigma, x)$ for each $x \in \Sigma$. The formula

$$v^N \;=\; \sum_{j=1}^{3}(v\cdot\tilde\tau_j)\tilde\tau_j$$

and the product rule give the pointwise estimate for the numerator,

$$|\nabla v^N| \;\le\; c\sum_{j=1}^{3}|\nabla\tilde\tau_j|\ , \tag{IV.36}$$

independent of the choice of $v \in \mathbb{S}^4$.

To estimate the denominator, we let v^L denote the orthogonal projection of v to any *fixed* 3 dimensional subspace L of $\mathbb{R}^5$, and observe the finiteness

$$C_1 \;=\; \int_{\mathbb{S}^4}\frac{1}{|v^L|}\, d\mathcal{H}^4 v \;<\; \infty\ ,$$

independent of L. To verify this, we note that the projection of $\mathbb{S}^4$ to L vanishes along a great circle, and, near any point of this circle, the projection is bilipschitz equivalent to an orthogonal projection of $\mathbb{R}^4$ to $\mathbb{R}^3$. So the finiteness of C_1 again follows from the finiteness of the 3 dimensional integral $\int_{\mathbb{R}^3\cap\mathbb{B}_1}|y|^{-1}\, dy$.

By Fubini's Theorem, (IV.35), (IV.36), and (IV.29),

$$\begin{aligned}
\int_{\mathbb{S}^4}\int_\Sigma |\nabla\sigma_1(x)|\, d\mathcal{H}^2 x\, d\mathcal{H}^4 v \;&\le\; 2\int_\Sigma |\nabla v^N(x)|\int_{\mathbb{S}^4}\frac{1}{|v^N(x)|}\, d\mathcal{H}^4 v\, d\mathcal{H}^2 x \\
&\le\; 2C_1\int_\Sigma |\nabla v^N(x)|\, d\mathcal{H}^2 x \\
&\le\; c\sum_{j=1}^{3}\int_\Sigma |\nabla\tilde\tau_j(x)|\, d\mathcal{H}^2 x \\
&\le\; c\int_{\mathbb{B}^5}|\nabla^2 u|^2\, dx\ .
\end{aligned}$$

So there exists a $v \in \mathbb{S}^4$ giving the σ_1 estimate

$$\int_\Sigma |\nabla\sigma_1(x)|\, d\mathcal{H}^2 x \;\le\; c\int_{\mathbb{B}^5}|\nabla^2 u|^2\, dx\ . \tag{IV.37}$$

Next we observe that $\sigma_2 = \frac{w_2}{|w_2|}$ where $w_2(x)$ is the orthogonal projection onto the 2 dimensional subspace $\mathrm{Nor}(\Sigma, x)\cap\sigma_1^\perp$. We again find

$$|\nabla\sigma_2| \;=\; |\nabla\left(\frac{w_2}{|w_2|}\right)| \;\le\; 2\frac{|\nabla w_2|}{|w_2|}\ . \tag{IV.38}$$

Now the formula

$$w_2 \;=\; \left[\sum_{j=1}^{3}(w\cdot\tilde\tau_j)\tilde\tau_j\right] \;-\; (w\cdot\sigma_1)\sigma_1\ ,$$

and the product rule give the pointwise estimate for the numerator,

$$|\nabla w_2| \ \le \ c \left(|\nabla \sigma_1| \ + \ \sum_{j=1}^{3} |\nabla \tilde{\tau}_j| \right) \ , \tag{IV.39}$$

independent of the choice $w \in \mathbb{S}^4$.

To estimate the denominator, we let w^M denote the orthogonal projection of w to any *fixed* 2 dimensional subspace M of the hyperplane $v^\perp = \sigma_1^\perp$, and observe the finiteness of the integral

$$C_2 \ = \ \int_{\mathbb{S}^4 \cap v^\perp} \frac{1}{|w^M|} \, d\mathcal{H}^3 w \ < \ \infty \ ,$$

independent of the choices of v or M. To verify this, we note that the projection of the 3 sphere $\mathbb{S}^4 \cap v^\perp$ to M vanishes along a great circle, where it is now bilipschitz equivalent to an orthogonal projection of $\mathbb{R}^3$ to $\mathbb{R}^2$. So the finiteness of C_2 this time follows from the finiteness of the 2 dimensional integral $\int_{\mathbb{R}^2 \cap \mathbb{B}_1} |y|^{-1} \, dy$.

By Fubini's Theorem, (IV.29), (IV.36), (IV.37), (IV.38) and (IV.39),

$$\begin{aligned}
\int_{\mathbb{S}^4 \cap v^\perp} \int_\Sigma |\nabla \sigma_2(x)| \, d\mathcal{H}^2 x \, d\mathcal{H}^3 w \quad &\le \quad 2 \int_\Sigma |\nabla w_2(x)| \int_{\mathbb{S}^4 \cap v^\perp} \frac{1}{|w_2(x)|} \, d\mathcal{H}^3 w \, d\mathcal{H}^2 x \\
&\le \quad 2 C_2 \int_\Sigma |\nabla w_2(x)| \, d\mathcal{H}^2 x \\
&\le \quad c \int_\Sigma \left(|\nabla \sigma_1(x)| \ + \ \sum_{j=1}^{3} |\nabla \tilde{\tau}_j(x)| \right) \, d\mathcal{H}^2 x \\
&\le \quad c \int_{\mathbb{B}^5} |\nabla^2 u|^2 \, dx \ .
\end{aligned}$$

So there exists a $w \in \mathbb{S}^4 \cap v^\perp$ giving the σ_2 estimate

$$\int_\Sigma |\nabla \sigma_2(x)| \, d\mathcal{H}^2 x \ \le \ c \int_{\mathbb{B}^5} |\nabla^2 u|^2 \, dx \ . \tag{IV.40}$$

Finally we may use the product rule and the formula

$$\begin{aligned}
\sigma_3 \quad = \quad & [(\sigma_1 \cdot \tilde{\tau}_2)(\sigma_2 \cdot \tilde{\tau}_3) - (\sigma_1 \cdot \tilde{\tau}_3)(\sigma_2 \cdot \tilde{\tau}_2)] \, \tilde{\tau}_1 \\
+ \quad & [(\sigma_1 \cdot \tilde{\tau}_3)(\sigma_2 \cdot \tilde{\tau}_1) - (\sigma_1 \cdot \tilde{\tau}_1)(\sigma_2 \cdot \tilde{\tau}_3)] \, \tilde{\tau}_2 \\
+ \quad & [(\sigma_1 \cdot \tilde{\tau}_1)(\sigma_2 \cdot \tilde{\tau}_2) - (\sigma_1 \cdot \tilde{\tau}_2)(\sigma_2 \cdot \tilde{\tau}_1)] \, \tilde{\tau}_3
\end{aligned}$$

along with (IV.29), (IV.37), and (IV.40) to obtain the σ_3 estimate

$$\int_\Sigma |\nabla \sigma_3(x)| \, d\mathcal{H}^2 x \ \le \ c \int_{\mathbb{B}^5} |\nabla^2 u|^2 \, dx \ . \tag{IV.41}$$

Now we may combine (IV.33), (IV.34), (IV.29), (IV.37), (IV.40), and (IV.41) to obtain the desired length estimate

$$\mathcal{H}^1(A) \ = \ \mathcal{H}^1(\rho_\Gamma \circ \gamma)^{-1}\{z\} \ \le \ c \int_{\mathbb{B}^5} |\nabla^2 u|^2 \, dx \ . \tag{IV.42}$$

IV.8 Connecting the Singularities b_j to $b_{j'}$

.

Although we now have a good description and length estimate for A, we are not done. The problem is that the set $\operatorname{Clos} A$ does not necessarily connect each of the original singularities a_i to another $a_{i'}$ or to $\partial\mathbb{B}^5$. The path in $\operatorname{Clos} A$ starting at a_i may end at some b_j. To complete the connections between pairs of a_i, it will be sufficient to find a *different* union B of curves which connect each frame singularity b_j to $\partial\mathbb{B}^5$ or to another unique frame singularity $b_{j'}$. Then adding to $\operatorname{Clos} A$ some components of B will give the desired curves connecting every a_i to a distinct $a_{i'}$ or to $\partial\mathbb{B}^5$. In this section we will use the map Φ from §IV.4 to construct this additional connecting set B, and we will, in §IV.9, obtain the required estimate on the length of B.

First we recall the description in [MS] of $\tilde{G}_2(\mathbb{R}^5)$ as a 2 sheeted cover of the Grassmannian of *unoriented* 2 planes in R^5. With $Q \in \tilde{G}_2(\mathbb{R}^5)$ chosen as before in §IV.3, consider the 5 dimensional Schubert cycle

$$\mathcal{S}_Q \;=\; \{P \in \tilde{G}_2(\mathbb{R}^5) \;:\; \dim\left(P \cap Q^\perp\right) \geq 1\}$$

and the 4 dimensional subcycle

$$\mathcal{T}_Q \;=\; \{P \in \tilde{G}_2(\mathbb{R}^5) \;:\; \dim\left(P \cap Q^\perp\right) \geq 2\} \;=\; \{P \in \tilde{G}_2(\mathbb{R}^5) \;:\; P \subset Q^\perp\}\ .$$

As in [MS], we see that $\mathcal{S}_Q \setminus \mathcal{T}_Q$ is a smooth embedded open 5 dimensional submanifold of $\tilde{G}_2(\mathbb{R}^5)$ and that $\tilde{G}_2(\mathbb{R}^5) \setminus \mathcal{S}_Q$ consists of two open 6 dimensional antipodal cells, D_+ centered at Q and D_- centered at $-Q$.

Next we will carefully define a (nearest-point) retraction map

$$\Pi_{\mathbb{Q}} \;:\; \tilde{G}_2(\mathbb{R}^5) \setminus \{Q, -Q\} \;\to \mathcal{S}_Q\ .$$

For $P \in D_+ \setminus \{Q\}$, there is a unique vector $v \in P \cap \mathbb{S}^4$ which is at maximal distance in $P \cap \mathbb{S}^4$ from $Q \cap \mathbb{S}^4$ and a unique vector w in $Q \cap \mathbb{S}^4$ that is closest to v; in particular, $0 < w \cdot v < 1$. Choose $A_P \in \mathrm{so}(5)$ so that the corresponding rotation $\exp A_P \in SO(5)$ maps w to v and maps $\tilde{w}$ to $\tilde{v}$ where $P = v \wedge \tilde{v}$ and $Q = w \wedge \tilde{w}$. Thus $\exp A_P$ maps Q to P, preserving orientation. Here $(\exp tA_P)(w)$ defines a geodesic circle in $\mathbb{S}^4$, and

$$t_P \;\equiv\; \inf\{t > 0 \;:\; w \cdot (\exp tA_P)(w) = 0\ \} \;>\; 1\ .$$

Then $(\exp 2t_pA_P)(w) = -w$ and $\exp 4t_pA_P = \mathrm{id}$. It follows that, in $\tilde{G}_2(\mathbb{R}^5)$, as t increases,

$$(\exp tA_P)(Q) \in D_+ \text{ for } \ 0 \leq t_P \quad\text{and}\quad (\exp tA_P)(Q) \in D_- \text{ for } \ t_P < t \leq 2t_P\ ,$$

$$(\exp 0A_P)(Q) = Q, \quad (\exp A_P)(Q) = P, \quad (\exp t_pA_P)(Q) \in \mathcal{S}_Q, \quad (\exp 2t_pA_P)(Q) = -Q\ ,$$

and we let $\Pi_Q(P) = (\exp t_pA_P)(Q)$.

As P approaches $\partial D_+ = \mathcal{S}_Q$, $t_P \downarrow 1$ and $|\Pi_Q(P) - P| \to 0$. Thus, let

$$\Pi_Q(P) \;=\; P \quad\text{for}\quad P \in \mathcal{S}_Q\ .$$

Also, let

$$\Pi_Q(P) \;=\; -\Pi_Q(-P) \quad\text{for}\quad P \in D_- \setminus \{-Q\}\ .$$

Next recall that the small tubular neighborhood V of the 4 dimensional subgrassmannian $\tilde{G}_2(\{0\}\times\mathbb{R}^4)$ was well-separated from Q. It follows that $\Pi_Q(V)$ is in a small neighborhood of the 4 dimensional cycle $\Pi_Q\left(\tilde{G}_2(\{0\} \times \mathbb{R}^4)\right)$. In particular, the 5 dimensional measure of $\Pi_Q(V)$ is small, and one easily finds $P \in \mathcal{S}_Q \setminus \Pi_Q(V)$ so that $p \notin \Pi_Q(W)$.

For $P \in \mathcal{S}_Q \setminus \mathcal{T}_Q$, the intersection $P \cap Q^\perp \cap \mathbb{S}^4$ consists of 2 antipodal points in $P \cap \mathbb{S}^4$ that are uniquely of maximal distance from $Q \cap \mathbb{S}^4$, and one sees that

$$\operatorname{Clos} \Pi_Q^{-1}\{P\}$$

contains a single semi-circular geodesic arc joining Q and $-Q$. For almost all $P \in \mathcal{S}_Q \setminus \mathcal{T}_Q$, this semi-circle meets transversely both Y, and, near $\pm Q$, each small surface

$$\Phi\left([\Sigma \cap \mathbb{B}_\delta(b_j)] \times \{e_j\} \right) .$$

We will choose $P \in \mathcal{S}_Q \setminus \mathcal{T}_Q$ also to be a regular value of $\Pi_Q \circ \Phi$. Since, near P, $\mathcal{S}_Q$ is is a smooth transverse (in fact, orthogonal) to $\Pi_Q^{-1}\{P\}$, we find, using IV.4, that the set

$$(\Pi_Q \circ \Phi)^{-1}\{P\} \;=\; \Phi^{-1}(\Pi_Q^{-1}\{P\})$$

is an embedded 1 dimensional submanifold, containing $\{(b_1, \pm e_1), \ldots, (b_m, \pm e_m)\}$. In small neighborhoods of any two points (b_j, e_j), $(b_j, -e_j)$ the set $\operatorname{Clos}(\Pi_Q \circ \Phi)^{-1}\{P\}$ consists of two smooth segments (antipodal in the $\mathbb{S}^4$ factor) which both project, under the projection

$$p_\Sigma \;:\; \Sigma \times \mathbb{S}^4 \;\to\; \Sigma \;,$$

onto a *single* segment in Σ which contains b_j. Continuing these two antipodal segments one direction in $(\Pi_Q \circ \Phi)^{-1}\{P\}$ gives antipodal paths whose final endpoints are $(b_{j'}, e_{j'})$, $(b_{j'}, -e_{j'})$ for some j' *distinct from* j. Here

$$e_{j'} \wedge e_{j'\Sigma} \;=\; \Phi\,(b_{j'}, \pm e_{j'}) \;=\; -\Phi\,(b_j, \pm e_j) \;=\; -e_j \wedge e_{j\Sigma} \;.$$

Composing either antipodal path with the projection p_Σ gives the same path connecting b_j and $b_{j'}$. Similarly, by continuing in the other direction and projecting gives Thus the whole set

$$B \;=\; p_\Sigma \left[(\Pi_Q \circ \Phi)^{-1}\{P\} \right]$$

provides the desired connection in Σ.

Also note that these two paths upstairs have similar orientations induced as fibers of the map $\Pi_Q \circ \Phi$. That is, in the notation of slicing currents [Fe], §4.3,

$$p_{\Sigma\#} \left\langle\, [\![\Sigma \times \mathbb{S}^4]\!]\,,\, \Pi_Q \circ \Phi\,,\, Q \,\right\rangle \;=\; 2(\mathcal{H}^2 \llcorner B) \wedge \vec{B} \;, \tag{IV.43}$$

where $\vec{B}$ is a unit tangent vectorfield along B (in the direction running from b_j to $b_{j'}$).

IV.9 Estimating the Length of the Connecting Set B

.

The definition of B depends on the choices of:

(1) the *point* $p \in \mathbb{S}^3$ near $\{0\} \times \mathbb{R}^4$ which gives the surface $\Sigma = u^{-1}\{p\}$ and the map

$$\Phi \;:\; (\Sigma \times \mathbb{S}^4) \setminus \mathcal{N}_\Sigma \;\to\; \tilde{G}_2(\mathbb{R}^5) \;, \quad \Phi(x, e) = e \wedge e_\Sigma(x) \;,$$

(2) the 2 *plane* $Q \in \tilde{G}_2(\mathbb{R}^5)$ near H which determines the retraction $\Pi_{\mathbb{Q}}$ of $\tilde{G}_2(\mathbb{R}^5) \setminus \{Q, -Q\}$ onto the 5 dimensional Schubert cycle $\mathcal{S}_Q$, and

(3) the 2 *plane* $P \in \mathcal{S}_Q \setminus \Pi_Q(V)$ which gives $B = p_\Sigma \left[(\Pi_Q \circ \Phi)^{-1}\{P\} \right]$. Having chosen $p \in \mathbb{S}^3$ as before to obtain estimate (IV.29), we need to chose Q and P to get the desired length estimate for B.

Concerning Q, we first readily verify that the retraction Π_Q is locally Lipschitz in $\tilde{G}_2(\mathbb{R}^5) \setminus \{Q, -Q\}$ and deduce the estimate

$$|\nabla \Pi_Q(S)| \ \leq \ \frac{c}{|S-Q||S+Q|} \quad \text{for} \quad S \in \tilde{G}_2(\mathbb{R}^5) \setminus \{Q, -Q\} \ . \tag{IV.44}$$

Using (IV.43) and [Fe], 4.3.1, we may integrate the slices to find that

$$\begin{aligned} \int_{\mathcal{S}_Q \setminus \Pi_Q(V)} p_{\Sigma\#} \langle [\![\Sigma \times \mathbb{S}^4]\!], \Pi_Q \circ \Phi, P \rangle \, d\mathcal{H}^5 P \quad &\leq \quad p_{\Sigma\#} \int_{\mathcal{S}_Q} \langle [\![\Sigma \times \mathbb{S}^4]\!], \Pi_Q \circ \Phi, P \rangle \, d\mathcal{H}^5 P \\ &= \quad p_{\Sigma\#} \left([\![\Sigma \times \mathbb{S}^4]\!] \llcorner (\Pi_Q \circ \Phi)^\# \omega_{\mathcal{S}_Q} \right) \ , \end{aligned}$$

where $\omega_{\mathcal{S}_Q}$ is the volume element of $\mathcal{S}_Q$. By (IV.43) and Fatou's Lemma,

$$\begin{aligned} \int_{\mathcal{S}_Q} 2\mathcal{H}^1 \left(p_\Sigma [(\Pi_Q \circ \Phi)^{-1}\{P\}] \right) \, d\mathcal{H}^5 P \quad &= \quad \int_{\mathcal{S}_Q} \mathbb{M}[p_{\Sigma\#} \langle [\![\Sigma \times \mathbb{S}^4]\!], \Pi_Q \circ \Phi, P \rangle] d\mathcal{H}^5 P \\ &\leq \quad \mathbb{M} \left[p_{\Sigma\#} \left([[\Sigma \times \mathbb{S}^4]\!] \llcorner (\Pi_Q \circ \Phi)^\# \omega_{\mathcal{S}_Q} \right) \right] \\ &= \quad \sup_{\alpha \in \mathcal{D}^1(\Sigma), |\alpha| \leq 1} \int_\Sigma \int_{\mathbb{S}^4} (\Pi_Q \circ \Phi)^\# \omega_{\mathcal{S}_Q} \wedge p_\Sigma^\# \alpha \ . \end{aligned} \tag{IV.45}$$

To estimate this last double integral, we recall from §IV.3 that, for each fixed $x \in \Sigma \setminus \{a_1, \ldots, a_m\}$,

$$\Phi(x, \cdot) \ : \ \mathbb{S}^4 \setminus \mathrm{Nor}(\Sigma, x) \ \to \ \mathcal{Q}_x \ \equiv \ \mathcal{Q}_{\mathrm{Tan}(\Sigma,x)} \setminus Y_x$$

is a the smooth, orientation-preserving, 2-sheeted cover map. Each map $\Phi(x, \cdot)$ depends only on $\mathrm{Tan}(\Sigma, x)$, and any two such maps are orthogonally conjugate. We will derive the formula

$$\left[(\Pi_Q \circ \Phi)^\# \omega_{\mathcal{S}_Q} \wedge p_\Sigma^\# \alpha \right] (x, \cdot) \ = \ \beta(x, \cdot) p_\Sigma^\# \omega_\Sigma(x) \wedge \Phi(x, \cdot)^\# \omega_{\mathcal{Q}_x} \tag{IV.46}$$

where ω_Σ and $\omega_{\mathcal{Q}_x}$ denote the volume elements of Σ and $\mathcal{Q}_x$ and $\beta(x, \cdot)$ is a smooth function on $\mathbb{S}^4 \setminus \mathrm{Nor}(\Sigma, x)$ satisfying

$$|\beta(x, e)| \ \leq \ \frac{c}{|\Phi(x,e) - Q|^5 |\Phi(x,e) + Q|^5} \sum_{j=1}^{3} |\nabla \tilde{\tau}_j(x)| \ \text{ for } \ e \in \mathbb{S}^4 \ . \tag{IV.47}$$

Before proving (IV.46), note that the decomposition on the righthand side is not necessarily smooth in x since the different $\mathcal{Q}_x$ may overlap for x near a critical point of $\Phi(\cdot, e)$ for some $e \in \mathbb{S}^4$. Nevertheless, the formula does imply the measurability of $\beta(x, e)$ in x, and so may be integrated over Σ.

To derive (IV.46), we first note that, with the factorization $\Sigma \times \mathbb{S}^4$, there are only two terms in the (p, q) decomposition of the 5 form,

$$(\Pi_Q \circ \Phi)^\# \omega_{\mathcal{S}_Q} \ = \ \Omega_{2,3} \ + \ \Omega_{1,4} \ .$$

Thus,

$$(\Pi_Q \circ \Phi)^\# \omega_{\mathcal{S}_Q} \wedge p_\Sigma^\# \alpha \ = \ 0 \ + \ \Omega_{1,4} \wedge p_\Sigma^\# \alpha \tag{IV.48}$$

because the term $\Omega_{2,3} \wedge p_\Sigma^\# \alpha$, being of type $(2+1, 3)$, must vanish.

For each $S = \Phi(x, \pm e) \in \mathcal{Q}_x \setminus Y_x$, we also have the factorization

$$\mathrm{Tan}(\tilde{G}_2(\mathbb{R}^5), S) \ = \ \mathrm{Nor}(\mathcal{Q}_x, S) \ \times \ \mathrm{Tan}(\mathcal{Q}_x, S) \ .$$

Let $\mu_1, \mu_2, \mu_3, \mu_4, \nu_1, \nu_2$ be an orthonormal basis of $\wedge^1 \mathrm{Tan}(\tilde{G}_2(\mathbb{R}^5), S)$ so that

$$\mu_1, \mu_2, \mu_3, \mu_4 \in \wedge^1 \mathrm{Tan}(\mathcal{Q}_x, S) \ , \ \nu_1, \nu_2 \in \wedge^1 \mathrm{Nor}(\mathcal{Q}_x, S) \ , \ \mu_1 \wedge \mu_2 \wedge \mu_3 \wedge \mu_4 = \omega_{\mathcal{Q}_x}(S) \ ;$$

thus, $0 = \nu_1(v) = \nu_2(v) = \mu_1(w) = \mu_2(w) = \mu_3(w) = \mu_4(w)$ whenever $v \in \mathrm{Tan}(\mathcal{Q}_x, S)$ and $w \in \mathrm{Nor}(\mathcal{Q}_x, S)$. We may expand the 5 covector

$$\begin{aligned} \Pi_Q^{\#}(\omega_{\mathcal{S}_Q})(S) \ = \ & \lambda_1 \, \nu_2 \wedge \mu_1 \wedge \mu_2 \wedge \mu_3 \wedge \mu_4 + \lambda_2 \, \nu_1 \wedge \mu_1 \wedge \mu_2 \wedge \mu_3 \wedge \mu_4 + \lambda_3 \, \nu_1 \wedge \nu_2 \wedge \mu_2 \wedge \mu_3 \wedge \mu_4 \\ & + \lambda_4 \, \nu_1 \wedge \nu_2 \wedge \mu_1 \wedge \mu_3 \wedge \mu_4 + \lambda_5 \, \nu_1 \wedge \nu_2 \wedge \mu_1 \wedge \mu_2 \wedge \mu_4 + \lambda_6 \, \nu_1 \wedge \nu_2 \wedge \mu_1 \wedge \mu_2 \wedge \mu_3 \end{aligned}$$

where

$$|\lambda_i| \leq \frac{c}{|S - Q|^5 |S + Q|^5} \ , \tag{IV.49}$$

by (IV.44). Applying $\Phi^{\#}$ (that is, $\wedge^1 D\Phi(x, e)$) to all covectors and taking the $(1, 4)$ component, we find that only the first two terms survive so that

$$\begin{aligned} \Omega_{1,4}(x, e) \ & = \ \left[\lambda_1 \Phi^{\#} \nu_2 + \lambda_2 \Phi^{\#} \nu_1\right]_{(1,0)} \wedge \Phi^{\#} \mu_1 \wedge \Phi^{\#} \mu_2 \wedge \Phi^{\#} \mu_3 \wedge \Phi^{\#} \mu_4 \\ & = \ \left[\lambda_1 \Phi^{\#} \nu_2 + \lambda_2 \Phi^{\#} \nu_1\right]_{(1,0)} \wedge \Phi(x, \cdot)^{\#} \omega_{\mathcal{Q}_x}(S) \ . \end{aligned} \tag{IV.50}$$

Being of type $(2, 0)$, the 2 covector

$$\left(\left[\lambda_1 \Phi^{\#} \nu_2 + \lambda_2 \Phi^{\#} \nu_1\right]_{1,0} \wedge p_{\Sigma}^{\#} \alpha \right) (x, e) \ = \ \beta(x, e) p_{\Sigma}^{\#} \omega_{\Sigma}(x) \tag{IV.51}$$

for some scalar $\beta(x, e)$, and (IV.48), (IV.50), and (IV.51) now give the desired formula (IV.46). This formula readily implies the smoothness of $\beta(x, \cdot)$ on $\mathbb{S}^4 \setminus \mathrm{Nor}(\Sigma, x)$.

To verify the bound (IV.47), observe that

$$|\left[\Phi^{\#} \nu_i\right]_{1,0}| \ = \ \sup_{v \in \mathbb{S}^4 \cap \mathrm{Tan}(\Sigma, x)} \nu_i [\nabla_v \Phi(x, e)] \ , \tag{IV.52}$$

where $\nabla_v \Phi(x, e) = D\Phi_{(x,e)}(v, 0) \in \mathrm{Tan}(\tilde{G}_2(\mathbb{R}^5), S)$. For any unit vector $v \in \mathrm{Tan}(\Sigma, x)$ and any $w \in \mathbb{R}^5$,

$$v \wedge w \in \mathrm{Tan}(\mathcal{Q}_x, S)$$

because we may assume $w \notin \mathrm{Tan}(\Sigma, x)$ and then choose a curve $y(t)$ in $\mathbb{S}^4 \cap v^{\perp} \setminus \mathrm{Nor}(\Sigma, x)$ with $y'(0) = w - (w \cdot v)v$, hence,

$$v \wedge w \ = v \wedge y'(0) \ = \ \frac{d}{dt}_{t=0} (v \wedge y(t)) \ = \ -\frac{d}{dt}_{t=0} \Phi\left(x, y(t)\right) \ .$$

Thus, for any 2 vector $\xi \in \mathrm{Nor}(\mathcal{Q}_x, S)$, $|\xi| \ = \ |\xi \wedge v|$; in particular, $|\xi| = |\xi \wedge \tilde{e}_T(x)|$, $|\xi| = |\xi \wedge e_{\Sigma}(x)|$, and hence,

$$|\xi| \ = | \, \xi \wedge (\tilde{e}_T(x) \wedge e_{\Sigma}(x)) \, | \ .$$

Since $\nu_i \in \wedge^1 \mathrm{Nor}(\mathcal{Q}_x, S)$ and $|\nu_i| = 1$, we now find that

$$\nu_i [\nabla_v \Phi(x, e)] \ = \nu_i \left[(\nabla_v \Phi(x, e))_{\mathrm{Nor}(\Sigma, x)} \right] \ \leq |\nabla_v \Phi(x, e) \wedge (\tilde{e}_T(x) \wedge e_{\Sigma}(x)) \, | \ . \tag{IV.53}$$

Moreover,

$$\begin{aligned} | \, \nabla_v \Phi \wedge (\tilde{e}_T \wedge e_{\Sigma})| \quad & \leq \quad | \, (\nabla_v (e \wedge e_{\Sigma})) \wedge (\tilde{e}_T \wedge e_{\Sigma})| \ \leq \ | \, (\nabla_v e_{\Sigma}) \wedge (\tilde{e}_T \wedge e_{\Sigma})| \\ & = \quad | \, e_{\Sigma} \wedge \nabla_v (\tilde{e}_T \wedge e_{\Sigma})| \ \leq \ |\nabla_v (\tilde{e}_T \wedge e_{\Sigma})| \\ & = \quad | \, \nabla_v \left(*(\, \tilde{\tau}_1 \wedge \tilde{\tau}_2 \wedge \tilde{\tau}_3) \right) \, | \ \leq \ c \textstyle\sum_{j=1}^3 |\nabla \tilde{\tau}_j| \ , \end{aligned} \tag{IV.54}$$

where $*$ is the Hodge $* : \wedge_3\mathbb{R}^5 \to \wedge_2\mathbb{R}^5 \approx \mathbb{R}^5$ [F,1.7.8]) The desired pointwise bound (IV.47) now follows by combining (IV.49), (IV.51), (IV.52), (IV.53) and (IV.54).

For each $x \in \Sigma$, the pull-back $\Phi(x,\cdot)^{\#}\omega_{\mathcal{Q}_x}$ is point-wise a positive multiple of the volume form of $\mathbb{S}^4$. So we may first integrate over $\mathbb{S}^4$ and use (IV.47) to see that

$$\begin{aligned}
\int_{\mathbb{S}^4} \beta(x,\cdot)\Phi(x,\cdot)^{\#}\omega_{\mathcal{Q}_x} &\le \int_{\mathbb{S}^4} |\beta(x,\cdot)|\Phi(x,\cdot)^{\#}\omega_{\mathcal{Q}_x} \\
&\le c\left(\sum_{j=1}^{3}|\nabla\tilde{\tau}_j(x)|\right)\int_{\mathbb{S}^4}\frac{\Phi(x,\cdot)^{\#}\omega_{\mathcal{Q}_x}}{|\Phi(x,\cdot)-Q|^5|\Phi(x,\cdot)+Q|^5} \\
&= c\left(\sum_{j=1}^{3}|\nabla\tilde{\tau}_j(x)|\right)\int_{\mathcal{Q}_x}\frac{\omega_{\mathcal{Q}_x}(S)}{|S-Q|^5|S+Q|^5} \\
&\le c\left(\sum_{j=1}^{3}|\nabla\tilde{\tau}_j(x)|\right)\int_{\mathcal{Q}_x}\frac{d\mathcal{H}^4 S}{|S-Q|^5|S+Q|^5}\ .
\end{aligned} \tag{IV.55}$$

To handle the denominator, we note that the Grassmannian $\tilde{G}_2(\mathbb{R}^5)$ is a 6 dimensional homogeneous space, and we readily use local coordinates to verify that

$$C_3 \;=\; \int_{\tilde{G}_2(\mathbb{R}^5)} \frac{1}{|S-Q|^5|S+Q|^5}\, d\mathcal{H}^6 Q \;<\; \infty\ , \tag{IV.56}$$

independent of S.

Now we recall (IV.45) and fix a sequence of 1 forms $\alpha_i \in \mathcal{D}^1(\Sigma)$ with $|\alpha_i| \le 1$ so that

$$\mathbb{M}\left[p_{\Sigma\#}\left(\llbracket\Sigma\times\mathbb{S}^4\rrbracket \llcorner (\Pi_Q\circ\Phi)^{\#}\omega_{\mathcal{S}_Q}\right)\right] \;=\; \lim_{i\to\infty}\int_\Sigma\int_{\mathbb{S}^4}(\Pi_Q\circ\Phi)^{\#}\omega_{\mathcal{S}_Q}\wedge p_\Sigma^{\#}\alpha_i\ ,$$

let β_i be the corresponding function from the formula (IV.46), and use (IV.45), Fatou's Lemma, (IV.46), (IV.55), Fubini's Theorem, (IV.56), and (IV.29) to obtain our final integral estimate

$$\begin{aligned}
&\int_{\tilde{G}_2(\mathbb{R}^5)}\int_{\mathcal{S}_Q} 2\mathcal{H}^1\left(p_\Sigma[(\Pi_Q\circ\Phi)^{-1}\{P\}]\right)\, d\mathcal{H}^5 P\, d\mathcal{H}^6 Q \\
&\le \int_{\tilde{G}_2(\mathbb{R}^5)} \lim_{i\to\infty}\int_\Sigma\int_{\mathbb{S}^4}(\Pi_Q\circ\Phi)^{\#}\omega_{\mathcal{S}_Q\setminus\Pi_Q(V)}\wedge p_\Sigma^{\#}\alpha_i \\
&\le \liminf_{i\to\infty}\int_{\tilde{G}_2(\mathbb{R}^5)}\int_\Sigma\int_{\mathbb{S}^4}\beta_i(x,\cdot)\omega_\Sigma(x)\wedge\Phi(x,\cdot)^{\#}\omega_{\mathcal{Q}_x} \\
&\le c\int_\Sigma\left(\sum_{j=1}^{3}|\nabla\tilde{\tau}_j(x)|\right)\int_{\mathcal{Q}_x}\int_{\tilde{G}_2(\mathbb{R}^5)}\frac{1}{|S-Q|^5|S+Q|^5}\, d\mathcal{H}^6 Q\; d\mathcal{H}^4 S\; d\mathcal{H}^2 x \\
&\le c\, C_3\sum_{j=1}^{3}\int_\Sigma|\nabla\tilde{\tau}_j(x)|\, d\mathcal{H}^2 x \quad\le\quad c\int_{\mathbb{B}^5}|\nabla^2 u|^2\, dx\ .
\end{aligned}$$

The Schubert cycles $\mathcal{S}_Q$ are all orthogonally equivalent and have the same positive 5 dimensional Hausdorff measure. So we can use the final integral inequality to choose first a 2 plane $Q \in \tilde{G}_2(\mathbb{R}^5)$ and then a 2 plane $P \in \mathcal{S}_Q$ so that the corresponding connecting set

$$B \;= p_\Sigma[(\Pi_Q\circ\Phi)^{-1}\{P\}]$$

satisfies the desired length estimate

$$\mathcal{H}^1(B) \ \leq \ c\int_{\mathbb{B}^5} |\nabla^2 u|^2\, dx \ .$$

■

Theorem IV.2 (Length Bound) *For any $u \in \mathcal{R}$, Sing u has a $\mathbb{Z}_2$ connection Γ satisfying*

$$\mathcal{H}^1(\Gamma) \ \leq \ c\int_{\mathbb{B}^5} |\nabla^2 u|^2\, dx \ ,$$

for some absolute constant c.

Proof. To form the connection Γ, one takes the union of the curves from A and B. The behavior of the individual curves near the points a_i and b_j has been discussed in subsection IV.5. The set $A \cup B$ will pass through each point b_j, likely having a corner at b_j. One easily replaces the corner with an embedded smooth curve near b_j. Also the curves contributing to A and B may cross. In $\mathbb{B}^5$, it is easy to perturb the curves to eliminate such crossings. The result is the desired $\mathbb{Z}_2$ connection Γ. (Alternately one can observe that $A \cup B$ already defines a one dimensional integer multiplicity chain modulo 2 (see [Fe][4.2.26] which has boundary $\sum_{i=1}^m [\![a_i]\!]$ relative to $\partial\mathbb{B}^5$. As mentioned before, the minimal (mass-minimizing) connection will automatcally consist of non-overlapping intervals. Those that reach $\partial\mathbb{B}^5$ meet it orthogonally.

V Sequential Weak Density of $W^{2,2}(\mathbb{B}^5, \mathbb{S}^3)$

We are now ready to prove:

Theorem V.3 *Any map v in $W^{2,2}(\mathbb{B}^5, \mathbb{S}^3)$ may be approximated in the $W^{2,2}$ weak topology by a sequence of smooth maps.*

Proof. First we may, by Lemma III.2, chose, for each positive integer i, a map $u_i \in \mathcal{R}$ so that $\|u_i - v\|_{W^{2,2}} < \frac{1}{i}$; in particular,

$$\|u_i - v\|_{L^2} \ < \ \frac{1}{i} \quad \text{and} \quad I \ = \ \sup_i \int_{\mathbb{B}^5} |\nabla^2 u_i|^2\, dx \ < \ \infty \ .$$

Applying Lemma III.1 to each u_i, we note that, as $\varepsilon \to 0$, the smooth approximates $u_{i,\varepsilon}$ approach u_i pointwise on $\mathbb{B}^5 \setminus \operatorname{Sing} u_i$. Inasmuch as the $u_{i,\varepsilon}$ are pointwise bounded (by 1), Lebesgue's theorem implies

$$\|u_{i,\varepsilon} - u_i\|_{L^2} \ \to \ 0 \quad \text{as} \quad \varepsilon \to 0 \ .$$

Thus we can choose a positive ε_i, so that the smooth map $w_i = u_{i,\varepsilon_i}$ has $\|w_i - u_i\|_{L^2} < \frac{1}{i}$; in particular, $\|w_i - v\|_{L^2} < \frac{2}{i}$, and the smooth maps w_i converge to v strongly in L^2.

On the other hand, by Lemma II.1 , Lemma III.1, Theorem IV.2, and (V),

$$\sup_i \|w_i\|^2_{W^{2,2}} \ < \ 2c_m(1 \ + \ I) \ < \ \infty \ .$$

By the weak*(=weak) compactness of the closed ball in $W^{2,2}(\mathbb{B}^5, \mathbb{R}^\ell)$, the sequence w_i contains a subequence $w_{i'}$ that is $W^{2,2}$ weakly convergent to some $w \in W^{2,2}(\mathbb{B}^5, \mathbb{R}^\ell)$. But, w, being by Rellich's theorem, the strong L^2 limit of the $w_{i'}$, must necessarily be the original map v. ■

V.1 Least Connection Length $L(v)$

By Lemma III.2, one may now define, for any Sobolev map $v \in W^{2,2}(\mathbb{B}^5, \mathbb{S}^3)$, the nonnegative number

$$L(v) \ = \ \lim_{\varepsilon \to 0} \inf\{\mathcal{H}^1(\Gamma) \ : \ \Gamma \text{ is a } \mathbb{Z}_2 \text{ connection for } \operatorname{Sing} u \text{ for some } u \in \mathcal{R} \text{ with } \|u - v\|_{W^{2,2}} < \varepsilon\} \ .$$

Any $u \in \mathcal{R}$ has a minimal $\mathbb{Z}_2$ connection, and $L(u)$ is its length. In general:

Theorem V.4 *For any $v \in W^{2,2}(\mathbb{B}^5, \mathbb{S}^3)$, $L(v) = 0 \iff v$ is the $W^{2,2}$ strong limit of smooth maps.*

Proof. The sufficiency is immediate from the definition of $L(v)$. To prove the necessity, we assume $L(v) = 0$. Then we may choose, for each i, a map $u_i \in \mathcal{R}$ along with a $\mathbb{Z}_2$ connection Γ_i of $\operatorname{Sing} u_i$ so that $\|u_i - v\|_{W^{2,2}} < 1/i$ and $\mathcal{H}^1(\Gamma_i) < 1/i$. As in the previous proof, there is an $\varepsilon_i < 1/i$ so that the smooth maps $w_i = u_{i,\varepsilon_i}$ converge strongly in L^2 and weakly in $W^{2,2}$ to v. The lower-semicontinuity

$$\int_{\mathbb{B}^5} |\nabla^2 v|^2 \, dx \ \leq \ \liminf_{i \to \infty} \int_{\mathbb{B}^5} |\nabla^2 w_i|^2 \, dx$$

follows. On the other hand, we have from Theorem III.1 the inequality

$$\int_{\mathbb{B}^5} |\nabla^2 w_i|^2 \, dx \ - \ \int_{\mathbb{B}^5} |\nabla^2 u_i|^2 \, dx \ \leq \ \frac{1}{i} + \frac{c_{\mathbb{SH}}}{i} \ .$$

as well as, from Lemma III.2, the $W^{2,2}$ strong convergence

$$\lim_{i \to \infty} \int_{\mathbb{B}^5} |\nabla^2 u_i|^2 \, dx \ = \ \int_{\mathbb{B}^5} |\nabla^2 v|^2 \, dx \ ,$$

which together imply the upper semi-continuity

$$\int_{\mathbb{B}^5} |\nabla^2 v|^2 \, dx \ \geq \ \limsup_{i \to \infty} \int_{\mathbb{B}^5} |\nabla^2 w_i|^2 \, dx.$$

The convergence of the total Hessian energies of the w_i to that of v, along with the $W^{2,2}$ weak convergence, now implies the $W^{2,2}$ strong convergence of the smooth maps w_i to v. ■

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