High-dimensional Helicities and rigidity of linked Foliations

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Abstract: We give an ergodic interpretation of Hopf-Novikov helicities as conjectured by V.I Arnold in [1]. We then extend to higher dimension the topological lower bounds obtained by M.Freedman and Z.X. He in [8] for energies of invariant forms of linked foliations.

I Introduction

I.1 Arnold's ergodic interpretation of the generalized Hopf Invariant and rigidity of knotted magnetic tubes

In their paper [7] and [8] M.H. Freedman and Z.-X. He consider the following problem : Let K be a knot in $\mathbb{R}^3$ (a regular embedding of S^1 in $\mathbb{R}^3$) and T be a regular tubular neighborhood of K in $\mathbb{R}^3$, one considers closed 2-forms ω in $\overline{T}$ such that the restriction of ω to the boundary of T is 0 (i.e. $\iota_{\partial T}^*\omega = 0$, where $\iota_{\partial T}$ is the inclusion map). They proved that for any such 2-form the following inequality holds

$$\frac{1}{4\pi}\int_T\int_T \frac{|\omega(x)|\ |\omega(y)|}{|x-y|^2} \geq [\mathrm{Flux}(\omega)]^2\ \ \mathrm{ac}(K) \tag{I.1}$$

where Flux(ω) is the integral of ω over any surface in $\overline{T}$ whose boundary lies in ∂T and whose intersection number with K is $+1$. Moreover ac(K) is the following knot invariant :

Let L be an embedded oriented closed curve in T, we denote by $\deg L$ the intersection number of L with any oriented section of T generating $H_2(T, \partial T; \mathbb{Z})$ (oriented such that the intersection number between Σ and K is $+1$). Then

$$\mathrm{ac}(K) = \inf\left\{ \frac{\mathrm{c}(L, L')}{|\deg L|\ |\deg L'|} \quad ; \quad L, L' \begin{array}{c}\text{closed emb.}\\ \text{curves in } T\end{array} \right\} \tag{I.2}$$

where $\mathrm{c}(L, L')$ is the over-crossing number of L and L' : the minimal number of over-crossing of L over L' among all planar knot diagrams representing (L, L'). The result above is also extended to the case of a general link (see [8]).

Since ω is a closed 2-form in dimension 3, in a neighborhood U of a point x_0 such that $\omega(x_0) \neq 0$, using Darboux theorem, ω can be written as $\omega = \phi^*(dx_1 \wedge dx_2)$, where $\phi : U \to D^2$.

In the very particular case where ω can be written <u>globally</u> as $\omega = \phi^*(dx_1 \wedge dx_2)$ where $\phi : T \to D^2$, the proof of (I.1) can be sketched as follows : Using Federer's coarea formula we have

$$\begin{aligned} &\frac{1}{4\pi}\int_{T\times T}\left|X_\phi(x)\wedge X_\phi(y).\frac{x-y}{|x-y|^3}\right|\ dx\ dy \\ &= \frac{1}{4\pi}\int_{D^2} d\sigma(\xi)\int_{D^2} d\sigma(\zeta)\int_{\phi^{-1}(\xi)\times\phi^{-1}(\zeta)}\left|\vec{t}(x)\wedge\vec{t}(y).\frac{x-y}{|x-y|^3}\right| \end{aligned} \tag{I.3}$$

where X_ϕ is the divergence free field associated to ω ($\iota_{X_\phi} dxdydz = \omega$ - ι is the interior product of vectors on forms - and this yields $X_\phi \in \mathrm{Ker}(\omega)$ and $< *\omega; X_\phi >= |\omega|^2$), $d\sigma$ is the volume form on D^2, $\vec{t}$ is the unit vector tangent to the preimages of regular points by ϕ and the measure on $\phi^{-1}(\xi) \times \phi^{-1}(\zeta)$ is the product measure obtained from the restriction of the ambiant metric of $\mathbb{R}^3$ to $\phi^{-1}(\xi)$ and $\phi^{-1}(\zeta)$. Consider two regular points of ϕ in D^2, ξ and ζ, for $(x, y) \in \phi^{-1}(\xi) \times \phi^{-1}(\zeta)$, $\left|\vec{t}(x) \wedge \vec{t}(y).\frac{x-y}{|x-y|^3}\right|$ is the norm of the pull-back of the volume form on S^2 : $d\mathrm{vol}_{S^2}$, by the map L

$$L \quad : \quad \phi^{-1}(\eta)\times\phi^{-1}(\xi) \longrightarrow S^2$$

$$(x,y) \longrightarrow \tfrac{x-y}{|x-y|}$$

Thus we have

$$\int_{\phi^{-1}(\xi)\times\phi^{-1}(\zeta)} \left|\vec{t}(x)\wedge\vec{t}(y).\frac{x-y}{|x-y|^3}\right| = \int_{s\in S^2} \mathrm{Card}(L^{-1}(s))\ d\mathrm{vol}_{S^2} \qquad \text{(I.4)}$$

For any regular value s of L, $\mathrm{Card}(L^{-1}(s))$ is the number of over-crossing of $\phi^{-1}(\eta)$ over $\phi^{-1}(\xi)$ for the projection on an oriented plane orthogonal to s. Thus (I.4) combined with the definition of $\mathrm{ac}(K)$ implies:

$$\frac{1}{4\pi}\int_{\phi^{-1}(\xi)\times\phi^{-1}(\zeta)} \left|\vec{t}(x)\wedge\vec{t}(y).\frac{x-y}{|x-y|^3}\right| \geq \mathrm{ac}(K)\ \deg\phi^{-1}(\xi)\ \deg\phi^{-1}(\eta) \quad \text{(I.5)}$$

Using once again the coarea formula, we have, for any generator Σ of $H_2(T,\partial T,\mathbb{Z})$ whose intersection number with K is $+1$,

$$\mathrm{Flux}(\phi^* d\sigma) = \int_\Sigma \phi^* d\sigma = \int_{D^2} d\sigma(\xi)\ \deg\phi^{-1}(\xi) \qquad \text{(I.6)}$$

Combining (I.3), (I.5) and (I.6) we establish (I.1) for such globally integrable form ω.

One of the main achievement of [8] is to extend this proof to the case where ω cannot be written globally as $\phi^* dx_1\wedge dx_2$. In such a situation one does not have anymore a nice indexation of the leaves, defined by ω, by the values of a map ϕ, the coarea formula cannot be applied and the decomposition of the Lebesgue measure on T as the measure $\frac{d\mathcal{H}^1}{|\omega|}$ along each leaf times some transversal measure, does not necessarily exists (in particular because a.e. leaf, for the transverse measure ω, is not necessarily compact and one would have to find a measure on the space of leaves that can be already particularly complicated, see [5]). The idea used to overcome this difficulty, originally appeared in a paper by V.I. Arnold [1], where he gives an ergodic description of what he called the asymptotic Hopf invariant. He introduced the flow g_t of the divergence free vector field X associated to ω (i.e. for all vector Y $<*\omega, Y>= (X,Y)$ where $<\ ,\ >$ means the duality between forms and vector and (,) is the scalar product in $\mathbb{R}^3$), this flow preserves the volume form and he replaced somehow the decomposition of the Lebesgue measure given by the coarea formula we had in the case $\omega = \phi^* d\sigma$, by the integration over the line flows of X :

$$\forall t > 0 \quad \forall f \in L^1(T)$$

$$\int_T f = \int_T \frac{1}{t} dx \int_0^t f(g_s(x))\ ds \tag{I.7}$$

This elementary identity means that, in order to integrate a function over the space T, one can start to take the mean value of f on the leaf starting at x up to $g_t(x)$ for the measure $dt = \frac{d\mathcal{H}^1}{|X|}$ and then, integrate these time averages over all of T relatively to the Lebesgue measure. Moreover, by the mean of Birkhoff Ergodic theorem, the time average in the integral of the right-hand-side of (I.7) converges almost everywhere. Precisely this idea was used in the following way. The Hopf invariant of a C^1 map u from S^3 into S^2 is given by

$$H(u) = \int_{S^3} u^*\omega \wedge \eta$$

where ω is any closed 2-form generating $H^2(S^2)$, normalized such that $\int \omega = 1$ and η is any 1-form such that $d\eta = u^*\omega$. The use of the coarea formula as above yields

$$H(u) = \frac{1}{4\pi^2} \int_{S^2} \int_{S^2} \mathrm{lk}(u^{-1}(\xi); u^{-1}(\zeta))\ d\xi\, d\zeta$$

where lk denotes the linking number between two curves of S^3. In [1] V.I. Arnold considers any closed 2-form on S^3 and the generalized the Hopf invariant also called the helicity of dA, introduced in hydrodynamics by H.K.Moffatt (see [15]),

$$\mathcal{H}(dA) = \int_{S^3} dA \wedge A$$

and he gave the following ergodic interpretation for $\mathcal{H}$

$$\mathcal{H}(dA) = \int_{S^3} \int_{S^3} \lambda(x; y)\ dx\, dy \tag{I.8}$$

where $\lambda(x; y)$ is the following average of linking numbers for a.e. (x, y)

$$\lambda(x; y) = \lim_{T \to +\infty} \frac{1}{T^2} \int_{s=0}^{T} \int_{t=0}^{T} \mathrm{lk}(\Gamma_t(x); \Gamma_s(y))\ dt\, ds \tag{I.9}$$

and Γ_t denotes the union of two paths $\Gamma_t(x) = G_t(x) \cup \Delta(x, g_t(x))$. $G_t(x)$ is the flow line of the divergence free vector associated to dA between x and $g_t(x)$ (g_t denotes the flow itself) and Δ is an assignment of a smooth curve $\Delta(x, y)$ to every couple (x, y) that connects the points x and y such that it depends on x and y in a measurable way and such that the integral of the Gauss-form α over the product of any two such curves or over the product of any such a curve with a $G_t(x)$ ($t \leq 1$) are uniformly bounded. Recall that the Gauss-form living in $\Lambda^2(\mathbb{R}^3 \times \mathbb{R}^3)$ is given by

$$\alpha = \sum_{i=1}^{3} \frac{1}{4\pi} \frac{x_i - y_i}{|x - y|^2} dx_{i+1} \wedge dy_{i-1}$$

where indices i are in $\mathbb{Z}_3$. Such assignment of curves Δ is called a "system of short paths" and it is proved that the limit λ is independent of the choice of such a Δ.

Combining this approach and the proof we gave above of (I.1) in the particular case where $\omega-$a.e leaves are compact, one can extend the lower bound to general ω (see [8]).

I.2 High dimensional helicities

At the end of [1], V.I. Arnold ask about the existence of a similar Ergodic interpretation like (I.8) but for the following higher dimensional helicities whose "compact" version was introduced by S.P.Novikov (see [16], [17]) in it's "geometric realization" of the theory of rational homotopy of D.Sullivan
.

Consider 2 closed 2-forms on S^4 dA and dB that are integrable

$$dA \wedge dA = 0 \qquad dB \wedge dB = 0 \tag{I.10}$$

and that commute

$$dA \wedge dB = 0 \tag{I.11}$$

then we introduce

$$\mathcal{N}(dA, dB) = \left(\int_{S^4} dA \wedge A \wedge B ; \int_{S^4} dB \wedge B \wedge A \right) \tag{I.12}$$

The "compact" version of these helicities arises as one computes the rational homotopy class of a map u from S^4 into $\mathbb{R}^3 \setminus \{x_1, x_2\}$ where x_1 and x_2 are

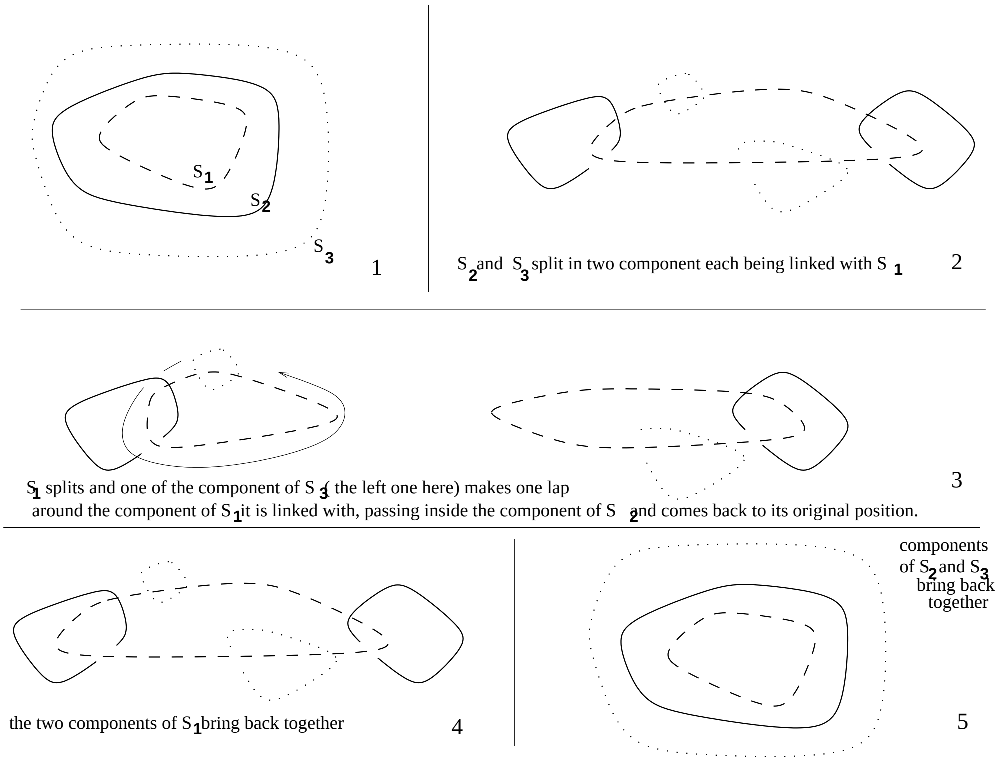

Figure 1: relative linking numbers of 3 torii S_1, S_2 and S_3 in $\mathbb{R}^4$: picture 1 to 5 represents 5 slicing increasing in time of $\mathbb{R}^4$, in that case we have $\mathrm{rlk}(S_3|S_2, S_1) = 1$ providing a right orientation of each torus - observe that $\mathrm{rlk}(S_2|S_3, S_1) = 0$.

two separated points of $\mathbb{R}^3$. The infinite part of $\pi_4(\mathbb{R}^3 \setminus \{x_1, x_2\})$ is $\mathbb{Z} \oplus \mathbb{Z}$ and the corresponding class to u is given by

$$\left(\int_{S^4} u^*\omega_1 \wedge \eta_1 \wedge \eta_2 ; \int_{S^4} u^*\omega_2 \wedge \eta_2 \wedge \eta_1 \right) \tag{I.13}$$

where $d\eta_1 = u^*\omega_1$, $d\eta_2 = u^*\omega_2$, and ω_1 and ω_2 are two 2-forms generating $H^2(\mathbb{R}^3 \setminus \{x_1, x_2\})$ and normalized such that $\int_{\sigma_i} \omega_i = 1$ (σ_i is any sphere around x_i).

In part I we give an interpretation of this integers in terms of the *relative linking number* of preimages by u of certain subparts of $\mathbb{R}^3 \setminus \{x_1, x_2\}$ (see proposition II.3). The *relative linking number* of a triplet $(\Sigma_0, \Sigma_1, \Sigma_2)$ of three closed disjoint surfaces of $\mathbb{R}^4$ is the topological degree of the following map

$$\begin{array}{rcl} V \quad : \quad \Sigma_0 \times \Sigma_1 \times \Sigma_2 & \longrightarrow & S^3 \times S^3 \\ & & \\ (x, y, z) & \longrightarrow & \left(\dfrac{x-y}{|x-y|}, \dfrac{x-z}{|x-z|} \right) \end{array} \tag{I.14}$$

$$\mathrm{rlk}(\Sigma_0 | \Sigma_1, \Sigma_2) := \deg(V) \tag{I.15}$$

See figure 1 for a non trivial relative linking number between 3 torii in $\mathbb{R}^4$.

Going back to the non-compact version of the Novikov helicities (I.12) we observe first that since $dA \wedge dA = 0$, Darboux theorem says that, in a neighborhood of a point where dA does not vanish, it may be written as $dA = \phi^* dx \wedge dy$ where ϕ is a submersion into D^2 (i.e. the kernel of dA is an integrable distribution of 2-planes). Thus dA defines a foliation in S^4 away from the set where dA vanishes. The leaf $\mathcal{L}^A(x)$ of this *almost foliation* defined by dA passing through $x \in S^4$, where $dA(x) \neq 0$, is the set

$$\begin{aligned} \mathcal{L}^A(x) = \Big\{ y \in S^4 \quad & \text{s.t. } \exists \gamma \in C^1([0,1]; \mathcal{T}_i) \quad \gamma(0) = x \ \gamma(1) = y \\ & \forall t \in [0,1] \quad dA_{\gamma(t)} \neq 0 \quad ; \quad \dot{\gamma}(t) \in \mathrm{Ker}(dA_{\gamma(t)}) \Big\} \end{aligned}$$

Let $\mathcal{X}$ be the set of oriented 2-submanifolds of S^4 which are everywhere transverse to the leaves, then dA defines a positive invariant measure over $\mathcal{X}$ (see [18]). We say that some property holds for dA-a.e. leaves of dA if for any N in $\mathcal{X}$ the measure relative to dA of the x in N such that the property does not hold for $\mathcal{L}^A(x)$ is 0.

Definition I.1 *The zero set of dA is said to be dA-negligible if for dA almost every leaf the zero set of dA has no intersection with the closure of the leaf. The foliation defined by dA is then called a lamination.*

Definition I.2 *A point x in S^4 is said to be essential for the lamination defined by dA if the leaf through x does not intersect the closure of $Ker\, dA$.*

We will assume from now on that both dA and dB define laminations.

We will need some "closing at infinity" for dA-a.e. leaf of dA. A leaf $\mathcal{L}^A(x)$ is said to be of Liouville type if there exists no non-constant harmonic function on $\mathcal{L}^A(x)$. For instance, a sufficient condition for beeing Liouville is, for the leaf $\mathcal{L}^A(x)$, to have subexponential growth (see [12]) i.e.

$$\forall y \in \mathcal{L}^A(x) \qquad \lim_{R\to+\infty} \frac{\log|B_R(y)|}{R} \longrightarrow 0$$

where $B_R(y)$ denotes the geodesic ball in $\mathcal{L}^A(x)$ of center y and radius R. The characterization that we will use of being of Liouville type for $\mathcal{L}$ is the cancellation of the Kaimanovich Entropy on $\mathcal{L}$ that was proved in [11] and [12] :

$$\lim_{t\to\infty} \frac{1}{t} \int_{\mathcal{L}} p(x,y,t)\ \log p(x,y,t)\ d\mathcal{H}^2(y) = 0 \tag{I.16}$$

for any x on $\mathcal{L}$ and where p is the Heat-Kernel for the laplacian on $\mathcal{L}$.

Definition I.3 *An integrable 2-form dA defines a Liouville lamination if it defines a lamination and if dA-a.e. leaf of this lamination is of Liouville type.*

Remark I.1 *Liouville laminations which are not necessarily compact are easy to produce. One takes a non degenerate closed two forms in a 2-torus contained $\mathbb{R}^3$ minus the Ox-axis. Assume this two form is tangent to the boundary of the torus and defines a non compact foliation inside. Then the non-compact Liouville lamination is obtained by rotating this foliation in $\mathbb{R}^4$ about the Ox axis.*

Consider now two 2-forms dA and dB verifying (I.10) and (I.11) each of these two defining a Liouville Lamination ((I.11) implies in particular that $dA\times dB-$a.e. pairs of $dA-$ and $dB-$leaves do not intersect - see lemma II.1). Let x be an essential point for dA we denote by p_A the heat-kernel on the leaf passing through x, $\mathcal{L}^A(x)$, for the metric induced on this leaf from $g_{S^4}/|dA|$. Let δ be any positive number, we define

$$\mathcal{L}^A_{k,\delta}(x,t) = \mathcal{L}^A(x) \cap \left\{\xi\ ;\ \delta^{k+1} \leq p_A(x,\xi,t) \leq \delta^k\right\}$$

and

$$\mathcal{D}^A_{k,\delta_T}(x,t) = \mathcal{L}^A_{k,\delta_T}(x,t) \cup C_{\partial\mathcal{L}^A_{k,\delta_T}(x,t)}$$

where C denotes an assignment of surfaces to any closed curve lying in a leaf of dA called a “system of small caps” (see proposition-definition II.1) replacing somehow the “system of short paths” Δ of the 1-D case. Let δ_T

be a positive function tending to 1 as $T \to +\infty$. We define Λ^T to be the following relative linking average

For (x, y, z) being an essential triplet for rep. dA , dA and dB

$\forall (k, l, m) \in \mathbb{N}^3$ let

$$\Lambda^T_{k,l,m}(x, y, z) = \frac{1}{T} \int_{[0,T]^3} \operatorname{rlk}\left(\mathcal{D}^A_{k,\delta_T}(x, t) | \mathcal{D}^A_{l,\delta_T}(y, s); \mathcal{D}^B_{m,\delta_T}(z, \sigma)\right) \; dt \, ds \, d\sigma$$

and define

$$\begin{aligned} &\Lambda^T(x, y, z) = \sum_{k,l,m} \delta_T^{k+l+m} \Lambda^T_{k,l,m}(x, y, z) \\ &\Lambda^T(x, y, z) = 0 \qquad \text{otherwise} \end{aligned} \tag{I.17}$$

We can state now our first main result

Theorem I.1 *Assume dA and dB are two integrable 2-forms of S^4 that commute, $dA \wedge dB = 0$, assume they both define Liouville laminations then the following ergodic interpretation of the Novikov helicity holds : there exists $\delta_T \to 1$ such that*

$$\lim_{T \to +\infty} \Lambda^T(x, y, z) = \Lambda(x, y, z) \qquad \textit{exists for a.e. } (x, y, z) \tag{I.18}$$

The limit is independent on the "system of small caps" chosen, moreover we have

$$\int_{S^4} dA \wedge A \wedge B = \int_{S^4} \int_{S^4} \int_{S^4} \Lambda(x, y, z) \tag{I.19}$$

An ergodic interpretation of the Hopf-Novikov helicities $\int_{S^4} dA \wedge A \wedge B$ was already obtained by B.A. Khesin in [13] and involves the notion of assymptotic linking between divergence free fields and foliations (see more about this approach in [14]). This ergodic interpretation is based on the following observation : since $dA \wedge dB = 0$ generically as two leaves $\mathcal{L}^A(x)$ and $\mathcal{L}^B$ intersect the resulting intersection is a line. The 1-dim "almost foliation" obtained is given by the flow of the divergence-free field associated to the form $dA \wedge B$. Then $\int dA \wedge B \wedge A$ is interpreted as a mean value of averages of linking numbers between the 1-dim lines of the foliation $dA \wedge B$ and the foliation given by dA. Again Arnolds approach via the flow along the divergence-free field associated to $dA \wedge B$ and the system of short path Δ is used but no ergodic interpretation of the linking number between a curve and the 2-dimensional "almost foliation" generated by dA is given. Using our approach, such an interpretation can be given in terms of averages of linking between curves and surfaces in the case where dA defines a Liouville lamination.

In particular our approach works also for the simplest helicity $\int dA \wedge A$, the Liouville restriction is not needed in this case since the leaves are 1-dimensional. The lamination condition was also required for the existence of short paths (see remark 4.14 page 145 of [2]) until the recent paper by T. Vogel [19].

Still for $M = \mathbb{R}^3 \setminus \{x_1, x_2\}$ one can iterate the procedure of computing the continuation algebras $C_q(A)$ of the minimal model A of $\wedge^* M$ (see [16] and [17]) in order to obtain all the geometric realizations of the rational homotopy class of $\mathbb{R}^3 \setminus \{x_1, x_2\}$. For the $\pi_4(\mathbb{R}^3 \setminus \{x_1, x_2\}) \otimes \mathbb{Q} \simeq \mathbb{Q}^2$ these geometric realizations are $u^*\omega_1 \wedge \eta_1 \wedge \eta_2$ and $u^*\omega_2 \wedge \eta_1 \wedge \eta_2$ that we saw before. One step further, for a map $u : S^5 \longrightarrow \mathbb{R}^3 \setminus \{x_1, x_2\}$, the geometric realization, induced by u, of the generators of the $H^5(C_5(A))$ gives the following 3 homotopic invariants corresponding to $\pi_5(\mathbb{R}^3 \setminus \{x_1, x_2\}) \otimes \mathbb{Q} \simeq \mathbb{Q}^3$: $u^*\omega_1 \wedge \eta_1 \wedge d^{-1}(u^*\omega_1 \wedge \eta_2)$, $u^*\omega_2 \wedge \eta_2 \wedge d^{-1}(u^*\omega_1 \wedge \eta_2)$ and $u^*\omega_1 \wedge \eta_2 \wedge d^{-1}(u^*\omega_1 \wedge \eta_2)$. The method we develop permits to give a geometric interpretation of these invariants in terms of higher order linkings with a corresponding Gauss integral formulas...etc. Moreover our method yields an ergodic interpretation of the following helicities in S^5 :

$$\int_{S^5} dB \wedge B \wedge C \quad \int_{S^5} dA \wedge A \wedge C \quad \int_{S^5} dA \wedge B \wedge C$$

where we assume that $A \in \wedge^1 S^5$, $B \in \wedge^1 S^5$ verify

$$dA \wedge dA = 0 \quad dB \wedge dB = 0 \quad dA \wedge dB = 0 \quad \text{ and } dC = dA \wedge B$$

These helicities are again just examples and we claim that the approach we develop for the helicity above (I.12) can be transposed to <u>all</u> the helicities one can construct from the Hopf-Novikov Invariants that give integral representation of the rational homotopy groups described in Sullivan Theory of the minimal models.

Passing from 1 dimensional leaves to 2 or any higher dimensional leaves required the introduction of a new way of "moving around over the leaves" compare to the one given by the flow of the divergence free field associated to the form in 3 dimension. This way of moving has to "decompose" the Lebesgue measure in the ambient space (like the coarea in the case of compact leaves) into a measure along the leaf times a measure over the set parameterizing the leaves. In the 1 dimensional case (I.7) the set of leaves was trivially over-parametrised by the all set of points of the ambient space. One of our main observation will be to interpret divergence free motion and the corresponding Birkhoff Ergodic Theorem as a motion on the leaf which is deterministic among motions of more general nature with more general ergodic theorem : we have

$$\int_{S^4} f = \int_{S^4} \frac{1}{t} \int_0^t d\tau \int_{Leaf} f(y) p(x, y, \tau) \tag{I.20}$$

where $p(x, y, \tau) = \delta_{\Phi_\tau(x)}$. Our idea is to replace this deterministic motion, which has no meaning for 2-dimensional leaves, by some Brownian motion on the leaves which is not deterministic anymore : we will look for a probability measure $p(x, dy, t)$, see theorem II.1, verifying the three following conditions

i) $p(x, dy, t)$ is supported on the leaf passing through x.

ii) If the leaf passing through x, $\mathcal{L}(x)$ is compact, for every function f

$$\lim_{t \to +\infty} \int_{\mathcal{L}(x)} f(y)\ p(x, dy, t) \longrightarrow c_{\mathcal{L}(x)} \int_{\mathcal{L}(x)} f(y) \frac{d\mathcal{H}^2}{|dA|}$$

iii) (*preservation of the Lebesgue measure*) For every function f

$$\int_{S^4} dx\ \frac{1}{t} \int_0^t d\tau \int_{S^4} f(y) p(x, dy, \tau) \longrightarrow \int_{S^4} f(x)\ dx$$

where dx denotes the Lebesgue measure. Moreover we will require to have an Ergodic property saying that the average in time in the left-hand side of the previous equality converges almost everywhere. We will view the introduction of such a decomposition of the Lebesgue measure as a substitute of the coarea formula in the case where the leaves are not necessarily compact and not necessarily indexed as preimages of points by some map.

I.3 Rigidity of linked Liouville Laminations

In this part we will apply the technics developed to solve theorem I.1 and to give the Ergodic interpretation of Hopf-Novikov's Invariants to get topological lower-bounds for conformal invariant energies of differential forms (such as the L^2 scalar product of 2-forms in 4 dimensions) defining Liouville Laminations in the spirit of estimate (I.1). To this aim we need to introduce few topological invariants.

Let Σ_1, Σ_2 and Σ_3 three closed disjoint (not necessarily connected) surfaces in $\mathbb{R}^4$ (or S^4), we define the *relative over-crossing number*, $\mathrm{rc}(\Sigma_1|\Sigma_2,\Sigma_3)$, of Σ_2 and Σ_3 relative to Σ_1 in the following way. For almost every vector u in S^3 the projection of Σ_2 and Σ_1 on a plane perpendicular to u are transverse to each other and for such a u we define the "shadow" of Σ_2 on Σ_1 to be the following set

$$S_u(\Sigma_1,\Sigma_2)=\left\{x\in\Sigma_1 \quad \text{s.t.} \quad \exists\, y\in\Sigma_2 \quad u=\frac{x-y}{|x-y|}\right\}$$

Take now another "generic" vector v of S^3 such that the projection of S_u and Σ_3 on a plane perpendicular to v are transverse to each other. The over-crossing set of S_u and Σ_3 relative to v is

$$S_{u,v}(\Sigma_1|\Sigma_2,\Sigma_3)=\left\{x\in S_u \quad \text{s.t.} \quad \exists\, z\in\Sigma_3 \quad v=\frac{x-z}{|x-z|}\right\}$$

The over-crossing number of Σ_2 and Σ_3 relative to Σ_1 is the minimum among all smooth deformations of $\mathbb{R}^4$ and among almost every "generic" u and v of the cardinal of $S_{u,v}$

$$\mathrm{rc}(\Sigma_1|\Sigma_2,\Sigma_3)=\min_{\text{a. e. } u,v\in S^3;\psi\in\text{ diffeo. }\mathbb{R}^4}\mathrm{Card}\left(S_{u,v}(\psi(\Sigma_1)|\psi(\Sigma_2),\psi(\Sigma_3))\right)$$

In the same spirit of [8] we can define the *asymptotic relative crossing number* of Σ_2 and Σ_3 relative to Σ_1, $\mathrm{arc}(\Sigma_1|\Sigma_2,\Sigma_3)$ to be the following number . We assume in a first approach that each of the 3 surfaces Σ_1, Σ_2 and Σ_3 are connected. Let $\mathcal{T}_1$, $\mathcal{T}_2$ and $\mathcal{T}_3$ be 3 disjoint tubular neighborhoods of Σ_1, Σ_2 and Σ_3 ($\mathcal{T}_i\simeq\Sigma_i\times D^2$) in $\mathbb{R}^4$. Inside $\mathcal{T}_1$, $\mathcal{T}_2$ and $\mathcal{T}_3$ respectively take Σ_1', Σ_2' and Σ_3' to be 3 closed surfaces (not necessarily connected) such that

$$\deg(\Sigma_i';\mathcal{T}_i):=[\Sigma_i']\cdot D_i\neq 0$$

where $[\Sigma'_i]$ is the homology class in $H_2(\mathcal{T}_i)$ of Σ'_i, D_i is the homology class in $H_2(\mathcal{T}_i, \partial)$ obtained from any section of $\mathcal{T}_i$ whose intersection number with $[\Sigma_i]$ is $+1$ ($\cdot$ is the intersection numbers operation). Then we define

$$\mathrm{arc}(\Sigma_1|\Sigma_2, \Sigma_3) =$$

$$\min \left\{ \frac{\mathrm{rc}(\Sigma'_1|\Sigma'_2, \Sigma'_3)}{\prod_{i=1}^3 |\deg(\Sigma'_i; \mathcal{T}_i)}| \quad \text{s. t.} \quad \begin{array}{c} \mathcal{T}_i \text{ disjoint tubu. neigh. of } \Sigma'_i \\ \\ \Sigma'_i \text{ clos. surfaces in } \mathcal{T}_i \end{array} \right\}$$

where the minimum is also taken among all smooth deformations of $\mathbb{R}^4$.

In the case where Σ_1, Σ_2 and Σ_3 are not connected anymore the definitions have to be changed. For instance, if Σ_1 and Σ_2 have 1 connected component each but Σ_3 has 2 connected components $\tilde{S}_3$ and $\hat{S}_3$, $\mathrm{arc}(\Sigma_1|\Sigma_2, \Sigma_3)$ is defined in the following way : for any choice of four surfaces Σ'_1, Σ'_2, $\tilde{\Sigma}'_3$ and $\hat{\Sigma}'_3$ in $\mathcal{T}_1$, $\mathcal{T}_2$, $\tilde{T}_3$ and $\hat{T}_3$ we consider the minimum among any generic u and v of the sum

$$\frac{S_{u,v}(\Sigma'_1|\Sigma'_2, \tilde{\Sigma}'_3)}{|\deg(\Sigma'_1, \mathcal{T}_1)\ \deg(\Sigma'_2, \mathcal{T}_2)\ \deg(\tilde{\Sigma}'_3, \mathcal{T}_3)|} + \frac{S_{u,v}(\Sigma'_1|\Sigma'_2, \hat{\Sigma}'_3)}{|\deg(\Sigma'_1, \mathcal{T}_1)\ \deg(\Sigma'_2, \mathcal{T}_2)\ \deg(\hat{\Sigma}'_3, \mathcal{T}_3)|}$$

and the minimum of such quantity among all possible choices of Σ'_1, Σ'_2, $\tilde{\Sigma}'_3$ and $\hat{\Sigma}'_3$ in $\mathcal{T}_1$, $\mathcal{T}_2$, $\tilde{T}_3$ and $\hat{T}_3$ and all possible smooth deformations gives $\mathrm{arc}(\Sigma_1|\Sigma_2, \Sigma_3)$.

Our second main result reads.

Theorem I.2 *Let $\mathcal{T}_1$, $\mathcal{T}_2$ and $\mathcal{T}_3$ be three disjoint tubular neighborhoods of three disjoint closed surfaces Σ_1, Σ_2 and Σ_3 in $\mathbb{R}^4$. Let dA_i be three integrable 2-forms ($dA_i \wedge dA_i = 0$) in $\mathcal{T}_i$ defining Liouville laminations and so that $\iota^*_{\partial\mathcal{T}_i} dA_i = 0$ ($\iota_{\partial\mathcal{T}_i}$ is the inclusion map of $\partial\mathcal{T}_i$ in $\mathbb{R}^4$) then the following identity holds*

$$\begin{array}{c} \displaystyle\frac{1}{16\pi^4} \int_{\mathcal{T}_1} \int_{\mathcal{T}_2} \int_{\mathcal{T}_3} \frac{|dA_1|(x)\ |dA_2|(y)\ |dA_3|(z)}{|x-y|^3\ |x-z|^3} \geq \\ \displaystyle\prod_{i=1}^3 |Flux(dA_i)| \quad arc(\Sigma_1|\Sigma_2, \Sigma_3) \end{array} \tag{I.21}$$

where $Flux(dA_i)$ is the integral over any oriented transverse section of $\mathcal{T}_i$ having intersection number $+1$ with Σ_i.

It is clear that

$$\operatorname{arc}(\Sigma_1|\Sigma_2,\Sigma_3) \geq |\operatorname{rlk}(\Sigma_1|\Sigma_2,\Sigma_3)|$$

Inequality (I.21) with $\operatorname{rlk}(\Sigma_1|\Sigma_2,\Sigma_3)$ instead of $\operatorname{arc}(\Sigma_1|\Sigma_2,\Sigma_3)$ does not require the dA_i to be integrable and follows by standard cohomological arguments using Poincaré duality (see for instance [3]) and the integral formula we give bellow for the *relative linking number*. In the last section we give an example of surfaces where

$$\operatorname{rlk}(\Sigma_1|\Sigma_2,\Sigma_3) = 0 \qquad \text{and} \qquad \operatorname{arc}(\Sigma_1|\Sigma_2,\Sigma_3) > 0$$

The *asymptotic crossing number* may be compared with other topological invariants : Consider any smooth 3-manifold M_2 bounding Σ_2 ($\partial M_2 = \Sigma_2$) and intersecting Σ_1 transversally. The intersection $M_2 \cap \Sigma_1$ defines an homology class $\sigma_{1,2}$ in $H_1(\mathcal{T}_1)$ which is independent on M_2 verifying $\partial M_2 = \Sigma_2$. This is nothing but the intersection pairing between the class defined by such an M_2 in $H_3(\mathbb{R}^4, \Sigma_2)$ and $H_2(\Sigma_1)$ (see [6] page 336). Taking now the restriction to $\mathbb{R}^4 \setminus \mathcal{T}_3$ of an immersed surface that bounds a smooth representant of $\sigma_{1,2}$, it defines a class $\sigma_{1,2,3}$ in $H_2(\mathbb{R}^4 \setminus \mathcal{T}_3, \Sigma_1 \cup \mathcal{T}_3)$ which is independent of the representant of σ_1 chosen (since $\Sigma_1 \cap \mathcal{T}_3 = \emptyset$). Observe that the boundary part of $\sigma_{1,2,3}$ in $H_1(\partial\mathcal{T}_3)$ intersected with M_3, a manifold bounding Σ_3, gives the relative linking number $\operatorname{rlk}(\Sigma_1|\Sigma_2,\Sigma_3)$. Then we prove

Theorem I.3 *Let $\sigma_{1,2,3}$ being the class in $H_2(\mathbb{R}^4\setminus\mathcal{T}_3; \Sigma_1\cup\partial\mathcal{T}_3)$ defined above, we have*

$$arc(\Sigma_1|\Sigma_2,\Sigma_3) \geq \|\sigma_{1,2,3}\|$$

where $\|\ \|$ denotes the singular pseudo-norm in $H_2(\mathbb{R}^4 \setminus \mathcal{T}_3; \Sigma_1 \cup \partial\mathcal{T}_3)$

$$\|\sigma_{1,2,3}\| = \inf \{1/n\ x(S)\ |\ f: S \to M \quad and$$

$$f_*[S] = n\sigma_{1,2,3}\ \ in\ H_2(\mathbb{R}^4 \setminus \mathcal{T}_3; \Sigma_1 \cup \partial\mathcal{T}_3)\}\,.$$

and $x(S) = \sum_i \chi_-(S_i)$ where S_i are the connected components of S, $\chi_-(S_i) = \max\{0, -\chi(S_i)\}$ and χ is the Euler characteristic.

II Preliminaries

II.1 The 4-dimensional Gauss formula for relative linkings

Let Σ_0, Σ_1 and Σ_2 be three closed surfaces in $\mathbb{R}^4$ such that $\Sigma_0 \cap \Sigma_1 = \emptyset$ and $\Sigma_0 \cap \Sigma_2 = \emptyset$. $\mathrm{rlk}(\Sigma_0|\Sigma_1, \Sigma_2)$ is the relative linking number of this triplet defined by (I.15). The following Gauss formula for the relative linking holds.

Proposition II.1 *We have*

$$rlk(\Sigma_0|\Sigma_1, \Sigma_2) = \int_{\Sigma_0 \times \Sigma_1 \times \Sigma_2} \mathcal{G} \tag{II.1}$$

where $\mathcal{G}$ is the following Gauss-form defined on $(\mathbb{R}^4)^3$

$$\mathcal{G} = \frac{1}{4\pi^4} \sum_{i \neq k} \sum_{j \neq l} \frac{x_i - y_i}{|x-y|^4} \frac{x_j - z_j}{|x-z|^4} dx_k\, dx_l \, * (dy_i\, dy_k) * (dz_j\, dz_l) \tag{II.2}$$

Proof of proposition II.1 Let $\Omega = \frac{1}{|S^3|} \sum_{i=1}^4 X_i\, (dX_1..dX_4)_i$ where we will use the notation $(dX_1..dX_4)_i := \iota_{e_i}(dX_1\, dX_2\, dX_3\, dX_4)$ and ι_{e_i} is the interior product with the i-th vector e_i of the canonical basis of $\mathbb{R}^4$. Ω is the volume form on S^3 normalized in such a way that $\int_{S^3} \omega = 1$. The integral representation of the topological degree of V defined by (I.14) is

$$\begin{aligned}\deg(V) &= \int_{\Sigma_0 \times \Sigma_1 \times \Sigma_2} V^*\Omega(X) \wedge \Omega(Y) \\ &= \frac{1}{4\pi^4} \int_{\Sigma_0 \times \Sigma_1 \times \Sigma_2} \sum_{i,j=1}^{4} \frac{x_i - y_i}{|x-y|^4} \frac{x_j - z_j}{|x-z|^4} \left(d(x_1 - y_1)..d(x_4 - y_4)\right)_i \\ &\quad \left(d(x_1 - z_1)..d(x_4 - z_4)\right)_j \end{aligned} \tag{II.3}$$

Few terms vanish while developing the product forms and the above integral becomes

$$\frac{1}{4\pi^4} \int_{\Sigma_0 \times \Sigma_1 \times \Sigma_2} \sum_{i \neq k} \sum_{j \neq l} \frac{x_i - y_i}{|x-y|^4} \frac{x_j - z_j}{|x-z|^4} dx_k\, dx_l\, (dy_1..dy_4)_{ik}\, (dz_1..dz_4)_{jl}$$

which gives the result.

The following interpretation of the relative linking number illustrates perhaps better the link between rlk and arc.

Proposition II.2 *Let Σ_0, Σ_1 and Σ_2 be three closed surfaces in $\mathbb{R}^4$ such that $\Sigma_0 \cap \Sigma_1 = \emptyset$ and $\Sigma_0 \cap \Sigma_2 = \emptyset$. Let M_1 and M_2 be two 3 submanifolds of $\mathbb{R}^4$ such that $\partial M_i = S_i$ (for $i = 1, 2$). Assuming M_2 and Σ_0 intersect each other transversally along a curve Γ_2 and that this curve intersects M_1 transversally then*

$$rlk(\Sigma_0|\Sigma_1, \Sigma_2) = \Gamma_2 \cdot M_1$$

where $\Gamma_2 \cdot M_1$ is the intersection number between Γ_2 and M_1.

Remark II.1 *It is not difficult to see that any Novikov integral expression of the rational homotopy class of a map from a sphere into a manifold (extending I.13 for $\pi_4(\mathbb{R}^3 \setminus \{p, q\}) \otimes \mathbb{Q}$ to arbitrary $\pi_k(N) \otimes \mathbb{Q}$) admits such an interpretation via intersection numbers between preimages of points and manifolds whose boundary are preimages of points. Such an interpretation makes a "geometric" illustration via intersection of submanifolds of the minimal model construction of Sullivan.*

Proof of proposition II.2 Let T_0, T_1 and T_2 be 3 disjoint tubular neighborhoods of the 3 disjoint surfaces Σ_0, Σ_1, Σ_2. Choose ϕ_i a smooth map from T_i into D^2 (the 2-unit disk of $\mathbb{R}^2$ centered at 0) such that $\phi(\partial T_i) \subset \partial D^2$ and the degree of ϕ_i on a given section S_i of T_i generating $H_2(T_i, \partial T_i)$ with intersection number $+1$ with Σ_i is $+1$. We have then $\iota^*_{\partial T_i} \phi^*_i \omega = 0$, where $\iota_{\partial T_i}$ is the canonical embedding of ∂T_i in T_i, and ω is the volume form on D^2 normalised so that $\int_{D^2} \omega = +1$. We have moreover $\int_{S_i} \phi^*_i \omega = 1$. From the homotopy invariance of degrees we get that for any arbitrary choice (p_0, p_1, p_2) of regular point of (ϕ_0, ϕ_1, ϕ_2) in $(D^2)^3$ $rlk(\phi_0^{-1}(p_0)|\phi_1^{-1}(p_1), \phi_2^{-1}(p_2))$ is independant of (p_0, p_1, p_2) and equals $rlk(\Sigma_0|\Sigma_1, \Sigma_2)$. We then obtain by the mean of the Gauss formula above and the coarea formula that

$$\begin{aligned} rlk(\Sigma_0|\Sigma_1, \Sigma_2) &= \int_{D^2}\int_{D^2}\int_{D^2} rlk(\phi_0^{-1}(p_0)|\phi_1^{-1}(p_1), \phi_2^{-1}(p_2))\ dp_0\, dp_1\, dp_2 \\ &= \int_{D^2}\int_{D^2}\int_{D^2} dp_0\, dp_1\, dp_2 \ \int_{\phi_0^{-1}(p_0)\times\phi_1^{-1}(p_1)\times\phi_2^{-1}(p_2)} \mathcal{G} \\ &= \int_{T_0\times T_1\times T_2} \phi_0^*\omega(x) \wedge \phi_1^*\omega(x) \wedge \phi_2^*\omega(x) \wedge \mathcal{G} \end{aligned} \tag{II.4}$$

Introducing η_1 and η_2 to be the following 1-forms

$$\eta_i := d^*\Delta^{-1}\phi_i^{-1}(p_1) = d^* \int_{x_i\in\mathbb{R}^4} \frac{1}{4\pi^2|x - x_i|^2} \phi_i^*\omega(x_i)$$

A simple computation yields that

$$\int_{T_0\times T_1\times T_2}\phi_0^*\omega(x)\wedge\phi_1^*\omega(x_1)\wedge\phi_2^*\omega(x_2)\wedge\mathcal{G}=\int_{\mathbb{R}^4}\phi_0^*\omega\wedge\eta_1\wedge\eta_2$$

It is a well known reslt that the η_i are the Poincaré duals in $H^1(\mathbb{R}^4\setminus T_i)$ of any cyle bounding Σ_i and the proposition follows.

II.2 Geometric interpretation of the Hopf-Novikov Invariants.

We give now the interpretation of the Hopf-Novikov Invariant in terms of relative linking.

Proposition II.3 *Let u be a regular map from S^4 into $\mathbb{R}^3\setminus\{p_1,p_2\}$ where p_1 and p_2 are two distinct points of $\mathbb{R}^3$. Let π_i $(i=1,2)$ be the orthogonal projection of $\mathbb{R}^3$ onto S_i^2 the unit-sphere of center p_i. Then for every regular pair of distinct points x,y of $u_1=\pi_1\circ u$ and every regular point z of $u_2=\pi_2\circ u$ verifying $\pi_1^{-1}(x)\cap\pi_2^{-1}(z)=\emptyset$ and $\pi_1^{-1}(y)\cap\pi_2^{-1}(z)=\emptyset$ in $\mathbb{R}^3\setminus\{p_1,p_2\}$, we have*

$$\int_{S^4}u^*\omega_1\wedge\eta_1\wedge\eta_2=rlk\left(u_1^{-1}(x)|u_1^{-1}(y);u_2^{-1}(z)\right)\tag{II.5}$$

where ω_i is any 2-form generating $H^2(\mathbb{R}^3\setminus\{p_i\})$ with the normalisation $\int_{\partial B_r(p_i)}\omega_i=\delta_{ij}$ (for sufficiently small r) and η_i is any form verifying $d\eta_i=u^*\omega_i$.

Proof of proposition II.3 We first replace S^4 with $\mathbb{R}^4$. If $\overline{\omega}_i$ denotes the unit volume-form on S_i^2, we take $\omega_i=\frac{1}{4\pi}\pi_i^*\overline{\omega}_i$ thus $u^*\omega_i=\frac{1}{4\pi}u_i^*\overline{\omega}_i$. We also take a particular choice for η_i such that $d\eta_i=u^*\omega_i$: Having in mind that $-\frac{1}{4\pi^2}\frac{1}{|x|^2}$ is the green kernel of the Laplacian on $\mathbb{R}^4$, we take

$$\eta_i=d^*\left[-\frac{1}{4\pi^2}\frac{1}{|x|^2}\star u^*\omega_i\right]$$

If we explicit the convolution $\star$ we get

$$
\begin{aligned}
\eta_i &= -\frac{1}{4\pi^2} d^* \left[\sum_{k<l} dx_k\, dx_l \int_{\mathbb{R}^4} \frac{\Omega_{\mathbb{R}^4}}{|x-y|^2} (u^*\omega_i, dy_k\, dy_l) \right] \\
&= \frac{1}{4\pi^2} \sum_{k<l} \sum_{j=1}^{4} *(dx_j \, *(dx_k\, dx_l)) \int_{\mathbb{R}^4} \frac{x_j - y_j}{|x-y|^4} * (dy_k\, dy_l) \wedge u^*\omega_i(y) \qquad \text{(II.6)} \\
&= \frac{1}{4\pi^2} \sum_{k \neq l} dx_k \int_{\mathbb{R}^4} \frac{x_l - y_l}{|x-y|^4} * (dy_k\, dy_l) \wedge u^*\omega_i(y)
\end{aligned}
$$

Substituting in the formula for $N(u)$ we get

$$
\begin{aligned}
&\int_{S^4} u^*\omega_1 \wedge \eta_1 \wedge \eta_2 \\
&= \frac{1}{16\pi^4} \sum_{i \neq k} \sum_{j \neq l} \int_{(\mathbb{R}^4)^3} \frac{x_i - y_i}{|x-y|^4} \frac{x_j - y_j}{|x-y|^4} \qquad \text{(II.7)} \\
&\qquad u^*\omega_1(x) \wedge dx_k\, dx_l \wedge u^*\omega_1(y) \wedge *(dy_i\, dy_k) \wedge u^*\omega_2(z) \wedge *(dz_j\, dz_l)
\end{aligned}
$$

The coarea formula of Federer tells us that $\forall F : \mathbb{R}^4 \to S^2$ and any 2-form α in $\mathbb{R}^4$ we have

$$
\int_{\mathbb{R}^4} \alpha \wedge F^*\Omega_{S^2} = \int_{\xi \in S^2} \Omega_{S^2}(\xi) \int_{F^{-1}(\xi)} \alpha
$$

Where Ω_{S^2} is the volume form on S^2. Applying the coarea formula three times in (II.7), writing $u^*\omega_i = u^*\overline{\omega}_i / 4\pi$ we obtain, using proposition II.1,

$$
\begin{aligned}
&\int_{S^4} u^*\omega_1 \wedge \eta_1 \wedge \eta_2 \\
&= \frac{1}{|S^2|^3} \int_{(S^2_1)^2 \times S^2_2} \mathrm{rlk}(u_1^{-1}(\xi) | u_1^{-1}(\zeta), u_2^{-1}(\nu))\ \omega_1(\xi) \wedge \omega_1(\zeta) \wedge \omega_2(\nu) \qquad \text{(II.8)}
\end{aligned}
$$

Now observe that the integral $\int_{S^4} u^*\omega_1 \wedge \eta_1 \wedge \eta_2$ is independent on the choices of the generators of $H^2(S^2)$ and is also independent on the choice of the η_i such that $d\eta_i = u^*\omega_i$.

Take ξ_0 and ζ_0 two regular points of u_1 and ν_0 a regular point of u_2. Replace now in the arguments above $\overline{\omega}_1(\xi)$, $\overline{\omega_1}(\zeta)$ and $\overline{\omega}_2(\nu)$ respectively by $\tilde{\omega}_1^{\xi_0}(\xi) = \frac{1}{|B_\sigma(\xi_0)|} \chi_{B_\sigma(\xi_0)} \overline{\omega}_1(\xi)$ $\tilde{\omega}_1^{\zeta_0}(\zeta) = \frac{1}{|B_\sigma(\zeta_0)|} \chi_{B_\sigma(\zeta_0)} \overline{\omega}_1(\zeta)$ and $\tilde{\omega}_2^{\nu_0}(\nu) = \frac{1}{|B_\sigma(\nu_0)|} \chi_{B_\sigma(\nu_0)} \overline{\omega}_2(\nu)$, where B_σ denotes a geodesic ball in S^2 of radius σ

and χ_{B_σ} is the characteristic function of this geodesic balls. We choose σ sufficiently small such that $\text{rlk}(u_1^{-1}(\xi)|u_1^{-1}(\zeta), u_2^{-1}(\nu))$ is independent of the choice of the triplet in $B_\sigma(\xi_0) \times B_\sigma(\zeta_0) \times B_\sigma(\nu_0)$ and then equal to $\text{rlk}(u_1^{-1}(\xi_0)|u_1^{-1}(\zeta_0), u_2^{-1}(\nu_0))$. So arguing like above we obtain.

$$\int_{S^4} u^*\omega_1 \wedge \eta_1 \wedge \eta_2 = \text{rlk}(u_1^{-1}(\xi_0)|u_1^{-1}(\zeta_0), u_2^{-1}(\nu_0)) \tag{II.9}$$

this proves proposition II.3.

II.3 Systems of "small caps"

we need the following elementary lemma.

Lemma II.1 *Let dA and dB be two 2-forms of S^4 such that*

$$dA \wedge dA = 0 \quad \textit{and} \quad dB \wedge dB = 0 \qquad \textit{in } S^4$$

assume they both define laminations $\mathcal{L}^A$ and $\mathcal{L}^B$. If they commute :

$$dA \wedge dB = 0$$

then for $dA-$a.e. x and for $dB-$a.e. y

$$\mathcal{L}^A(x) \cap \mathcal{L}^B(y) = \emptyset$$

Proof of lemma II.1 Assume this is not the case, then we may find two saturated subsets $\mathcal{S}^A$ and $\mathcal{S}^B$ of $\mathcal{L}^A$ and $\mathcal{L}^B$ of non zero dA and dB measures such that $\forall x \in \mathcal{S}^A$ and $\forall y \in \mathcal{S}^B$

$$\mathcal{L}^A(x) \cap \mathcal{L}^B(y) \neq \emptyset$$

Because of the lamination hypothesis, we may also take $\mathcal{S}^A$ and $\mathcal{S}^B$ to be subparts of S^4 where respectively dA and dB do not degenerate with $|dA| \geq c > 0$ on $\mathcal{S}^A$ and $|dB| \geq c > 0$ on $\mathcal{S}^B$. Then, since $dA \wedge dB = 0$, dB restricted to $\mathcal{L}^A(x)$ defines a 1-foliation. Then there exists an at most countable union of 1-dimensional curves $(\Gamma_n)_{n\in\mathbb{N}}$ in $\mathcal{L}^A(x)$ transverse to this foliation such that every leaf of this 1-dimensional foliation crosses $\Gamma = \cup_n \Gamma_n$. Since every $dB-$leaf of $\mathcal{S}^B$ is assumed to intersect $\mathcal{L}^A(x)$, every $dB-$leaf of $\mathcal{S}^B$ intersects $\Gamma = \cup_n \Gamma_n$ that has $dB-$measure 0. So $\mathcal{S}^B$ has $dB-$ measure 0 which is a contradiction.

Proposition-Definition II.1 *Let dA_1, dA_2 and dA_3 be 3 closed 2-forms on S^4 such that*

$$\forall i,j=1,2,3 \qquad dA_i\wedge dA_j=0$$

Assume that the dA_i define laminations. Denote by $\mathcal{L}^{A_i}(x)$ the leaf of dA_i that passes through x. Then for $dA_1\times dA_2\times dA_3-$ a.e. leaves $\mathcal{L}^{A_1}(x_1)\times\mathcal{L}^{A_2}(x_2)\times\mathcal{L}^{A_3}(x_3)$ the following property holds : for any triplet of subsets K_1, K_2 and K_3 of $\mathcal{L}^{A_1}(x_1)$, $\mathcal{L}^{A_2}(x_2)$ and $\mathcal{L}^{A_3}(x_3)$ such that the curvatures of ∂K_1, ∂K_2 and ∂K_3 are uniformly bounded we may assign 3 union of embedded 2-disks C_{K_1}, C_{K_2} and C_{K_3} verifying

-
$$\forall i=1,2,3 \qquad |C_{K_i}|\le C|\partial K_i|$$

-
$$\left|\int_{C_{K_1}}\int_{C_{K_2}}\int_{C_{K_3}}\mathcal{G}\right|\le C|\partial K_1|\ |\partial K_2|\ |\partial K_3|$$

- $\forall i=1,2,3$
$$\left|\int_{K_i}\int_{C_{K_{i+1}}}\int_{C_{K_{i-1}}}\mathcal{G}\right|\le C|K_i|\ |\partial K_{i+1}|\ |\partial K_{i-1}|$$

- $\forall i=1,2,3$
$$\left|\int_{K_i}\int_{K_{i+1}}\int_{C_{K_{i-1}}}\mathcal{G}\right|\le C|K_i|\ |K_{i+1}|\ |\partial K_{i-1}|$$

where $\mathcal{G}$ is the Gauss-form given by (II.2). where the constant C may depend on the x_i but not on the K_i in $\mathcal{L}^{A_i}(x_i)$. A choice of such an assignment is called a system of "small caps".

Proof of proposition II.1 : For convenience of the presentation we give the proof in the case where the dA_i's are supported in a compact sub-domain of $\mathbb{R}^4$. Because of the previous lemma, for $dA_1\times dA_2\times dA_3$ a.e. triplet (x_1,x_2,x_3) $\forall i\neq j$, $\mathcal{L}^{A_i}(x_i)\cap\mathcal{L}^{A_j}(x_j)=\emptyset$. Moreover since they define Laminations, for $dA_1\times dA_2\times dA_3$ a.e. triplets (x_1,x_2,x_3) $|dA_i|\ge c>0$ on $\mathcal{L}^{A_i}(x_i)$ for $i=1,2,3$ and then the three leaves $\mathcal{L}^{A_i}(x_i)$ for $i=1,2,3$ have bounded geometries. Take such a triplet. We then may find a Lipschitz diffeomorphism of $\mathbb{R}^4$ such that the restriction to the 3 skeleton of a given lattice $L_\delta = \delta\mathbb{Z}^4$ of $\mathbb{R}^4$ (for a sufficiently small size δ) of the 3 leaves are made of flat segments. Since all of the ∂K_i have uniformly bounded curvature, we may modify K_i a bit, keeping it's area proportional to the original one and the length of ∂K_i proportional to it's original one, in order to ensure that ∂K_i lies in the 3-skeleton of L_δ keeping K_i in it's leaf $\mathcal{L}^{A_i}(x_i)$ and also to ensure that ∂K_i is made of a union of straight segments such that each connected component of ∂K_i restricted to any 3-cell is made exactly of 1 segment. Take now one component of ∂K_i denote it by l_i. l_i admits a projection $\hat{l}_i$ in the 1-skeleton of $\delta\mathbb{Z}^4$ such that the area of the annulus a_i bounding $l_i \cup \hat{l}_i$ is proportional to the length of l_i, moreover $|l_i| \simeq |\hat{l}_i|$. Solve now the plateau problem for $\hat{l}_i$ and denote by d_i a minimal disk that bounds $\hat{l}_i$. Since $\hat{l}_i$ lies in a compact part of $\mathbb{R}^4$ we clearly have $|d_i| \leq C|\hat{l}_i|$. We project now d_i in the 2-skeleton of L_δ in the following way : first we project d_i in the 3-skeleton using the following argument. Let c_δ be a given 4-cell of L_δ, we claim that we can choose a point p in the interior of c_δ such that the radial projection π_p from c_δ onto ∂c_δ relative to p keeps the area of the projection of $d_i \cap c_\delta$ proportional to the area of $d_i \cap c_\delta$ itself. Indeed let v be the conformal map from the unit disk solving our Plateau problem for $\hat{l}_i$, we have

$$\text{area}(c_\delta \cap d_i) = \frac{1}{2} \int_{v^{-1}(c_\delta \cap d_i)} |\nabla v(x)|^2 \, dx$$

Simple computations show that

$$\text{area}(\pi_p(c_\delta \cap d_i)) \leq C \int_{v^{-1}(c_\delta \cap d_i)} \frac{|\nabla v|^2}{|v(x) - p|^2} \, dx$$

integrating over p in the half 4-cube $c_\delta/2$ we get

$$\begin{aligned}
&\int_{p \in c_\delta/2} \text{area}(\pi_p(c_\delta \cap d_i)) \, dp \leq C \int_{p \in c_\delta/2} \int_{v^{-1}(c_\delta \cap d_i)} \frac{|\nabla v|^2}{|v(x) - p|^2} \, dx \, dp \\
&\leq C \int_{v^{-1}(c_\delta \cap d_i)} |\nabla v|^2
\end{aligned}$$

applying the mean-value formula we get such a p. Then using the same idea we can project $\pi_p(c_\delta \cap d_i)$ onto the 2-cell of L_δ and we then obtain a disk $\hat{d}_i$ that bounds $\hat{l}_i$ and which is made of flat pieces of the 2-skeleton of L_δ and such that the number of pieces is bounded by $C|l_i|/\delta$. We take C_{K_i} to be the union of these disks for the various components of ∂K_i. The reason why ii), iii) and iv) hold comes from the fact that the integral of the Gauss-form over a triplet of 2-parralelograms of size δ is bounded independently from their relative position in 4-space by a constant depending on δ.

II.4 Leaf harmonic measures and ergodic theorem for leaf-heat diffusions on laminations

First of all we prove the following key observation.

Proposition II.4 *Let dA be a closed 2-form of S^4 that is integrable : $dA \wedge dA = 0$. Assume that dA defines a lamination $\mathcal{L}^A$. Denote by Ω_{S^4} the standard volume-form on S^4. Take the metric on S^4 to be $g^A = g_{S^4}/|dA|$. Then Ω_{S^4} is g^A-leaf-harmonic for the lamination defined by dA :*

$$\forall \phi \in C^\infty(S^4) \quad \forall \mathcal{S} \quad \text{measurable non deg. saturated set of } \mathcal{L}^A$$

$$\int_{\mathcal{S}} \Delta_{\mathcal{L}^A}\phi \ \Omega_{S^4} = 0$$

where $\Delta_{\mathcal{L}^A}$ denotes the Laplace-Beltrami operator restricted to the leaves of $\mathcal{L}^A$.

Proof of proposition II.4 : Let $\mathcal{S}$ be a non degenerated measurable saturated set : $|dA| \geq c > 0$ on $\mathcal{S}$, for every $x \in \mathcal{S}$ $\mathcal{L}^A(x) \subset \mathcal{S}$. Take a ϕ in $C^\infty(S^4)$, we may always assume that ϕ has a support contained in a sufficiently small ball $B_r(x)$ of center x such that we can apply Darboux Theorem in the whole ball and we have on $B_r(x)$ $dA = H^*dx_1 \wedge dx_2$ where H is a map from $B_r(x)$ into D^2 (indeed since dA defines a lamination, $\mathcal{S}$ is contained in a compact set where $|dA| > 0$ and we can extract a finite covering from any covering of balls where Darboux theorem applies for dA and construct a partition of unity from this covering). Applying the Coarea Formula to H we have

$$\begin{aligned}
\int_{\mathcal{S}} \Delta_{\mathcal{L}^A}\phi \ \Omega_{S^4} &= \int_{\xi \in D^2 \cap H(\mathcal{S})} \int_{H^{-1}(\xi)} \Delta_{\mathcal{L}^A}\phi \ \frac{d\mathcal{H}^2}{|dA|} \\
&= \int_{\xi \in D^2 \cap H(\mathcal{S})} \int_{H^{-1}(\xi)} \Delta_{\mathcal{L}^A}\phi \ \mathrm{dvol}_{\mathcal{L}^A}
\end{aligned}$$

where $d\mathrm{vol}_{\mathcal{L}^A}$ is the volume form on the leaves induced by the metric $g_{S^4}/|dA|$. (Observe that $H^{-1}(\xi)$ is a portion of leaf from our lamination). So clearly

$$\int_{H^{-1}(\xi)} \Delta_{\mathcal{L}^A}\phi \ d\mathrm{vol}_{\mathcal{L}^A} = 0$$

and the proposition II.4 is proved.

Denote by $p^A(x,y,t)$ the heat-kernel for the Laplace Beltrami operator $\Delta_{\mathcal{L}^A}$ on the leaf $\mathcal{L}^A(x)$. Since the leaf is contained on a compact set where $|dA| > 0$ it has bounded geometry for the metric induced by $g_{S^4}/|dA|$. So the leaf is complete for the diffusion and from [4] we have

$$p^A(x,y,t) > 0 \quad \text{and} \int_{\mathcal{L}^A(x)} p^A(x,y,t) \ d\mathrm{vol}_{\mathcal{L}^A(x)}(y) = 1$$

$p^A(x,dy,t) = p^A(x,y,t) \ d\mathrm{vol}_{\mathcal{L}^A}(y)$ defines a probability measure in S^4 in the following way

$$p^A(x,E,t) = \int_{E\cap\mathcal{L}^A(x)} p^A(x,y,t) \ d\mathrm{vol}_{\mathcal{L}}(y)$$

This probability measure verifies the Chapman-Kolmogorov equation

$$p^A(t+s,x,E) = \int_{S^4} p^A(x,dy,t) \ p^A(y,E,s)$$

We claim now that the standard volume form on S^4 is invariant under this diffusion : for any measurable non degenerate saturated set $\mathcal{S}$ we have

$$\forall E \text{ measurable} \qquad \int_{\mathcal{S}} p^A(x,E,t)\Omega_{S^4}(x) = \int_{\mathcal{S}\cap E} \Omega_{S^4} \tag{II.10}$$

or equivalently for any f measurable in $\mathcal{S}$

$$\int_{\mathcal{S}} \Omega_{S^4}(x) \int_{\mathcal{S}} p^A(x,y,t) \ f(y) \ d\mathrm{vol}_{\mathcal{L}^A}(y) = \int_{\mathcal{S}} f(x) \ \Omega_{S^4}(x) \tag{II.11}$$

This result can be deduced from the theorem of L. Gardnett in [9] which says that being leaf harmonic for a measure implies it's invariance under the diffusion of the corresponding heat kernel. Formally the proof can be sketched like this : for any $f \in C^\infty$ we have, using the definition of the heat Kernel

$$\left[\frac{\partial}{\partial t} + \Delta_{\mathcal{L}}\right] \left(\int_{\mathcal{S}} p^A(x, y, t)\ f(y)\ d\mathrm{vol}_{\mathcal{L}}(y)\right) = 0$$

Using the fact that Ω_{S^4} is leaf harmonic, taking v to be $v = \int_{\mathcal{S}} p^A(x, y, t)\ f(y)\ d\mathrm{vol}_{\mathcal{L}}(y)$ and the characterization of leaf harmonicity of Ω_{S^4} applied to that v (proposition II.4) yields

$$\int_{\mathcal{S}} \Omega_{S^4}\ \Delta_{\mathcal{L}^A} \left(\int_{\mathcal{S}} p^A(x, y, t)\ f(y)\ d\mathrm{vol}^A_{\mathcal{L}}(y)\right) = 0$$

Combining the two previous identities we obtain

$$\frac{d}{dt}\left[\int_{\mathcal{S}} \Omega_{S^4} \left(\int_{\mathcal{S}} p^A(x, y, t)\ f(y)\ d\mathrm{vol}_{\mathcal{L}}(y)\right)\right] = 0$$

and since $\int_{\mathcal{S}} p^A(x, y, 0)\ f(y)\ d\mathrm{vol}_{\mathcal{L}}(y) = f$ we get that Ω_{S^4} is invariant under the diffusion.

Thus $p^A(x, E, t)$ defines a Markov process with the Ω_{S^4} measure as an invariant measure. Adapting the result by Yosida (see [21]) to our situation we deduce the following ergodic theorem

Theorem II.1 *Let dA be a closed integrable 2-form on S^4 (i.e. $dA \wedge dA = 0$). Assume that dA defines a lamination. Denote by $\mathcal{S}$ a measurable saturated set for dA and by $p^A(x, y, t)$ the heat-kernel for the metric induced by $g_{S^4}/|dA|$ on every leaf of dA (g_{S^4} is the standard metric on S^4). Then for any f in $L^1(\mathcal{S}; \Omega_{S^4})$ the following limit exists almost everywhere*

$$\lim_{T \to +\infty} \frac{1}{T} \int_0^T dt \int_{\mathcal{S}} f(y) p^A(x, dy, t) = f_\star$$

and is in L^1. Moreover we have

$$\int_{\mathcal{S}} f_\star\ \Omega_{S^4} = \int_{\mathcal{S}} f\ \Omega_{S^4}$$

III Proof of theorem I.1

Let dA and dB be two integrable closed 2-forms of S^4 which commute ($dA \wedge dB = 0$). and assume that they both define laminations : the zero sets of

dA and dB are respectively dA and dB-negligeable. For the convenience of the presentation we assume that both dA and dB are defined on a compact subset of $\mathbb{R}^4$. From the lamination hypothesis it is not difficult to deduce the existence of saturated subsets $\mathcal{S}_\varepsilon^A$ and $\mathcal{S}_\varepsilon^B$ for respectively dA and dB such that

- $|dA| \geq c_\varepsilon > 0$ on $\mathcal{S}_\varepsilon^A$ and $|dB| \geq c_\varepsilon > 0$ on $\mathcal{S}_\varepsilon^B$.

-
$$\left|\int_{\mathcal{S}_\varepsilon^A\times\mathcal{S}_\varepsilon^A\times\mathcal{S}_\varepsilon^B}\mathcal{G}^{A,A,B}(x,y,z)-\int_{\mathbb{R}^4\times\mathbb{R}^4\times\mathbb{R}^4}\mathcal{G}^{A,A,B}(x,y,z)\right|\leq\varepsilon$$

where

$$\begin{aligned}&\mathcal{G}^{A,A,B}(x,y,z)\\&=\frac{1}{4\pi^4}\sum_{i\neq k}\sum_{j\neq l}\frac{x_i-y_i}{|x-y|^4}\frac{x_j-y_j}{|x-y|^4}\\&\quad dA(x)\wedge dx_k\,dx_l\wedge dA(y)\wedge *(dy_i\,dy_k)\wedge dB(z)\wedge *(dz_j\,dz_l)\end{aligned}\tag{III.1}$$

Using theorem II.1 we know the existence of the limit

$$\begin{aligned}&\lim_{T\to+\infty}\frac{1}{T^3}\int_{[0,T]^3}\int_{\mathcal{S}_\varepsilon^{A,A,B}}p^A(x,d\xi,t)\;p^A(y,d\zeta,s)\;p^B(z,d\nu,\sigma)\;g^{AAB}(\xi,\zeta,\nu)\\&=\Lambda^\star(x,y,z)\end{aligned}\tag{III.2}$$

for a.e. $(x,y,z)\in\mathcal{S}_\varepsilon^A\times\mathcal{S}_\varepsilon^A\times\mathcal{S}_\varepsilon^B$ where

$$g^{AAB}(\xi,\zeta,\nu)=\left\langle\mathcal{G}^{A,A,B};\Omega^3_{\mathbb{R}^4}\right\rangle(\xi,\zeta,\nu)$$

and that

$$\int_{\mathcal{S}_\varepsilon^{A,A,B}}\Lambda^\star\;\Omega^3_{\mathbb{R}^4}=\int_{\mathcal{S}_\varepsilon^{A,A,B}}\mathcal{G}^{A,A,B}\tag{III.3}$$

where $\mathcal{S}_\varepsilon^{A,A,B}$ denotes $\mathcal{S}_\varepsilon^A\times\mathcal{S}_\varepsilon^A\times\mathcal{S}_\varepsilon^B$, and p^A and p^B are respectively the heat kernels for the heat operator on the leaves of dA and dB for the metrics induced by $g_{\mathbb{R}^4}/|dA|$ and $g_{\mathbb{R}^4}/|dB|$. Moreover $p^A(x,d\xi,t)$ denotes the distribution form $p^A(x,\xi,t)d\mathrm{vol}_{\mathcal{L}^A(x)}(\xi)$ and $d\mathrm{vol}_{\mathcal{L}^A(x)}$ is the volume form of the dA-leaf passing by x induced by $g_{\mathbb{R}^4}/|dA|$ whose associated measure on the leaf is $d\mathcal{H}^2/|dA|\lfloor\mathcal{L}^A$ ($d\mathcal{H}^2$ is the two-dimensional Hausdorff measure). We modify p^A and p^B in the following way. Let $\delta_T > 0$ to be fixed later on in the proof, we just assume now that $\delta_T \to 1$ as $T \to +\infty$ and we omit the subscript T. Using the mean value formula we get that for every integer k there exists $\delta^{k+1} < p < \delta^k$ such that

$$\begin{aligned} |\{p^A(x,\xi,t) = p\}| &= \int_{\{\xi\ ;\ p^A(x,\xi,t)=p\}} \frac{d\mathcal{H}^1}{|dA|^{\frac{1}{2}}(\xi)} \\ &\leq \frac{1}{1-\delta}\frac{1}{\delta^k}\int_{\delta^{k+1}}^{\delta^k} dq \int_{\{\xi\ ;\ p^A(x,\xi,t)=q\}} \frac{d\mathcal{H}^1}{|dA|^{\frac{1}{2}}(\xi)} \\ &\leq \frac{1}{1-\delta}\frac{1}{\delta^k}\int_{\mathcal{L}^A(x)\cap\{\xi\ ;\ \delta^{k+1}\leq p^A(x,\xi,t)\leq\delta^k\}} |\nabla_\xi p|\ \mathrm{dvol}_{\mathcal{L}^A(x)}(\xi) \end{aligned} \tag{III.4}$$

where we used the coarea formula. Choose such a p, denote it by $p^A_{k,\delta}(x,t)$ and let

$$\mathcal{L}^A_{k,\delta}(x,t) = \left\{\xi \in \mathcal{L}^A(x) \quad \text{s. t.} \quad p^A_{k+1,\delta}(x,t) \leq p^A(x,\xi,t) \leq p^A_{k,\delta}(x,t)\right\}$$

Denote by $\tilde{p}^A(x,\xi,t)$ the approximated kernel such that

$$\tilde{p}^A(x,\xi,t) = p^A_{k,\delta}(x,t) \qquad \text{on} \ \ \mathcal{L}^A_{k,\delta}(x,t)$$

By construction we have

$$\forall \xi \in \mathcal{L}^A(x) \quad \forall t > 0 \qquad \left|\frac{p^A(x,\xi,t) - \tilde{p}^A(x,\xi,t)}{p^A(x,\xi,t)}\right| \leq 1 - \frac{1}{\delta} \tag{III.5}$$

keeping in mind that δ tends to 1 as $T \to +\infty$ so that the relative difference between p^A and $\tilde{p}^A$ will be small. Precisely we have, omitting to explicitly write x, y, z and t, s, σ,

$$p^A(d\xi) = \tilde{p}^A(d\xi) + \left[\frac{p^A - \tilde{p}^A}{p^A}\right](\xi)\ p^A(d\xi)$$

$$p^A(d\zeta) = \tilde{p}^A(d\zeta) + \left[\frac{p^A - \tilde{p}^A}{p^A}\right](\zeta)\ p^A(d\zeta)$$

and

$$p^B(d\nu) = \tilde{p}^B(d\nu) + \left[\frac{p^B - \tilde{p}^B}{p^B}\right](\nu)\ p^B(d\nu)$$

Clearly, from (III.5) we deduce

$$\int_{\mathcal{S}_\varepsilon^{A,A,B}} \left| \int_{\mathcal{S}_\varepsilon^{A,A,B}} \left[\frac{\tilde{p}^A - p^A}{p^A} \right] g^{AAB}\ p^A(d\xi)\ p^A(d\zeta)\ p^B(d\nu) \right|$$
$$\leq (1-\frac{1}{\delta}) \int_{\mathcal{S}_\varepsilon^{A,A,B}} \int_{\mathcal{S}_\varepsilon^{A,A,B}} p^A(d\xi)\ p^A(d\zeta)\ p^B(d\nu) \frac{|dA|(\xi)\ |dA|(\zeta)\ |dB|(\nu)}{|\xi-\zeta|^3 |\xi-\nu|^3}$$

Taking the time average of it and the limit as $T \to +\infty$, we have

$$\lim_{T\to+\infty} \int_{\mathcal{S}_\varepsilon^{A,A,B}} \frac{1}{T^3} \int_{[0,T]^3} \left| \int_{\mathcal{S}_\varepsilon^{A,A,B}} \left[\frac{\tilde{p}^A - p^A}{p^A} \right] (\xi)\ p^A(d\xi)\ p^A(d\zeta)\ p^B(d\nu)\ g^{AAB} \right|$$
$$\leq \lim_{T\to+\infty} (1-\frac{1}{\delta_T}) \int_{\mathcal{S}_\varepsilon^{A,A,B}} \frac{|dA|(x)\ |dA|(y)\ |dB|(z)}{|x-y|^3 |x-z|^3} = 0 \tag{III.6}$$

Thus for a.e. (x, y, z) in $S_\varepsilon^{A,A,B}$ we have

$$\lim_{T\to+\infty} \frac{1}{T^3} \int_{[0,T]^3} \int_{\mathcal{S}_\varepsilon^{A,A,B}} \left[\frac{\tilde{p}^A - p^A}{p^A} \right] (\xi)\ p^A(d\xi)\ p^A(d\zeta)\ p^B(d\nu)\ g^{AAB} = 0 \tag{III.7}$$

This result can be extended in a similar way to all the other error terms when we replace p by $\tilde{p}$ in (III.2), so that we obtain for a.e. (x, y, z) in $S_\varepsilon^{A,A,B}$

$$\lim_{T\to+\infty} \frac{1}{T^3} \int_{[0,T]^3} \int_{\mathcal{S}_\varepsilon^{A,A,B}} \tilde{p}^A(x, d\xi, t)\ \tilde{p}^A(y, d\zeta, s)\ \tilde{p}^B(z, d\nu, \sigma)\ g^{AAB}(\xi, \zeta, \nu)$$
$$= \Lambda^\star(x, y, z) \tag{III.8}$$

and

$$\int_{\mathcal{S}_\varepsilon^{A,A,B}} \mathcal{G}^{A,A,B} = \int_{\mathcal{S}_\varepsilon^{A,A,B}} \lim_{T\to+\infty} \frac{1}{T^3} \int_{[0,T]^3} \int_{\mathcal{S}_\varepsilon^{A,A,B}} g^{AAB}\ \tilde{p}^A(d\xi)\ \tilde{p}^A(d\zeta)\ \tilde{p}^B(d\nu) \tag{III.9}$$

We decompose now the leaves along the sets $\mathcal{L}_{k,\delta}^A$, $\mathcal{L}_{l,\delta}^A$ and $\mathcal{L}_{m,\delta}^B$ where the $\tilde{p}^A$ and $\tilde{p}^B$ are constant, and we obtain

$$\int_{\mathcal{S}_\varepsilon^{A,A,B}} g^{AAB}\ \tilde{p}^A(d\xi)\ \tilde{p}^A(d\zeta)\ \tilde{p}^B(d\nu) =$$
$$\sum_{k,l,m} \delta^{k+l+m} \int_{\mathcal{L}_{k,\delta}^A(x,t)} \int_{\mathcal{L}_{l,\delta}^A(y,s)} \int_{\mathcal{L}_{m,\delta}^B(z,\sigma)} g^{AAB}\ dvol_{\mathcal{L}^A(x)}\ dvol_{\mathcal{L}^A(y)}\ dvol_{\mathcal{L}^B(z)} \tag{III.10}$$

Recall that the volumes are taken with respect to the metrics $g^A = g_{\mathbb{R}^4}/|dA|$, $g^A = g_{\mathbb{R}^4}/|dA|$ and $g^B = g_{\mathbb{R}^4}/|dB|$. We claim now that the restriction of $\mathcal{G}^{A,A,B}$ to $\mathcal{L}^A(x)\times\mathcal{L}^A(y)\times\mathcal{L}^B(z)$ coincide with $g^{AAB}\, dvol_{\mathcal{L}^A(x)}\, dvol_{\mathcal{L}^A(y)}\, dvol_{\mathcal{L}^B(z)}$. Indeed it suffices to observe that for any i and j in $\{1...4\}$, we have

$$|\langle dA\wedge dx_i\, dx_j; \Omega_{\mathbb{R}^4}\rangle| = |dA||i^*_{\mathcal{L}^A} dx_i\, dx_j| \tag{III.11}$$

where $i_{\mathcal{L}^A}$ is the isometric embedding of $\mathcal{L}^A$ into $\mathbb{R}^4$ and $|\quad|$ denotes the scalar product on 2-forms induced by the canonical scalar product in $\mathbb{R}^4$, $g_{\mathbb{R}^4}$. Taking into account the orientation and the fact that $g^A = g_{\mathbb{R}^4}/|dA|$ we have

$$\langle dA\wedge dx_i\, dx_j; \Omega_{\mathbb{R}^4}\rangle = \langle i^*_{\mathcal{L}^A} dx_i\, dx_j; dvol_{\mathcal{L}^A}\rangle \tag{III.12}$$

So then

$$\begin{aligned}\langle dA\wedge dx_i\, dx_j; \Omega_{\mathbb{R}^4}\rangle\, dvol_{\mathcal{L}^A} &= \langle i^*_{\mathcal{L}^A} dx_i\, dx_j; dvol_{\mathcal{L}^A}\rangle\ dvol_{\mathcal{L}^A}\\ &= i^*_{\mathcal{L}^A} dx_i\, dx_j\end{aligned} \tag{III.13}$$

Thus

$$\begin{aligned}&\int_{\mathcal{L}^A_{k,\delta}(x,t)}\int_{\mathcal{L}^A_{l,\delta}(y,s)}\int_{\mathcal{L}^B_{m,\delta}(z,\sigma)} g^{AAB}\ dvol_{\mathcal{L}^A(x)}\ dvol_{\mathcal{L}^A(y)}\ dvol_{\mathcal{L}^B(z)}\\ &=\int_{\mathcal{L}^A_{k,\delta}(x,t)\times\mathcal{L}^A_{l,\delta}(y,s)\times\mathcal{L}^B_{m,\delta}(z,\sigma)} \mathcal{G}\end{aligned} \tag{III.14}$$

where $\mathcal{G}$ is the Gauss-form introduced in proposition II.1. We need now to close the $\mathcal{L}^A_{k,\delta}(x,t)$, the $\mathcal{L}^A_{l,\delta}(y,s)$ and the $\mathcal{L}^B_{m,\delta}(z,\sigma)$ by the mean of the system of "small caps" introduced in proposition II.1. Let $\mathcal{D}^A_{k,\delta}(x,t) = \mathcal{L}^A_{k,\delta}(x,t)\cup C_{\partial\mathcal{L}^A_{k,\delta}(x,t)}$...etc, where $C_{\partial\mathcal{L}^A_{k,\delta}(x,t)}$ is the union of small caps for the connected components of $\partial\mathcal{L}^A_{k,\delta}(x,t)$ given by proposition II.1. Using the result in this proposition we deduce that

$$\left|\int_{\mathcal{L}^A_{k,\delta}(x,t)\times\mathcal{L}^A_{l,\delta}(y,s)\times\mathcal{L}^B_{m,\delta}(z,\sigma)}\mathcal{G}-\int_{\mathcal{D}^A_{k,\delta}(x,t)\times\mathcal{D}^A_{l,\delta}(y,s)\times\mathcal{D}^B_{m,\delta}(z,\sigma)}\mathcal{G}\right|\le$$
$$\le C\ |\partial\mathcal{L}^A_{k,\delta}(x,t)|\ |\mathcal{L}^A_{l,\delta}(y,s)|\ |\mathcal{L}^B_{m,\delta}(z,\sigma)|+|\mathcal{L}^A_{k,\delta}(x,t)|\ |\partial\mathcal{L}^A_{l,\delta}(y,s)|\ |\mathcal{L}^B_{m,\delta}(z,\sigma)|$$
$$+C\ |\mathcal{L}^A_{k,\delta}(x,t)|\ |\mathcal{L}^A_{l,\delta}(y,s)|\ |\partial\mathcal{L}^B_{m,\delta}(z,\sigma)|+|\partial\mathcal{L}^A_{k,\delta}(x,t)|\ |\partial\mathcal{L}^A_{l,\delta}(y,s)|\ |\mathcal{L}^B_{m,\delta}(z,\sigma)|$$
$$+C\ |\partial\mathcal{L}^A_{k,\delta}(x,t)|\ |\mathcal{L}^A_{l,\delta}(y,s)|\ |\partial\mathcal{L}^B_{m,\delta}(z,\sigma)|+|\mathcal{L}^A_{k,\delta}(x,t)|\ |\partial\mathcal{L}^A_{l,\delta}(y,s)|\ |\partial\mathcal{L}^B_{m,\delta}(z,\sigma)|$$
$$+C\ |\partial\mathcal{L}^A_{k,\delta}(x,t)|\ |\partial\mathcal{L}^A_{l,\delta}(y,s)|\ |\partial\mathcal{L}^B_{m,\delta}(z,\sigma)| \tag{III.15}$$

Multiplying the quantities in the right-hand side of (III.15) by δ^{k+l+m} and summing over k, l and m, using (III.4), we get for the first term for instance

$$\begin{aligned}&\sum_{k,l,m}\delta^{k+l+m}|\partial\mathcal{L}^A_{k,\delta}(x,t)|\ |\mathcal{L}^A_{l,\delta}(y,s)|\ |\mathcal{L}^B_{m,\delta}(z,\sigma)|\\ &\le\frac{1}{(1-\delta)\delta^3}\sum_{k,l,m}\int_{\tilde{\mathcal{L}}^A_{k,\delta}(x,t)}|\nabla_\xi p^A|\int_{\mathcal{L}^A_{l,\delta}(y,s)}p^A(y,d\zeta,s)\int_{\mathcal{L}^B_{m,\delta}(z,\sigma)}p^B(z,d\nu,\sigma)\\ &\le\frac{1}{(1-\delta)\delta^3}\int_{\mathcal{L}^A(x)}|\nabla_\xi p^A|dvol_{\mathcal{L}^A}\end{aligned} \tag{III.16}$$

where $\tilde{\mathcal{L}}^A_{k,\delta}(x,t)=\mathcal{L}^A(x)\cap\{\xi\ ;\ \delta^{k+1}\le p^A(\xi)\le\delta^k\}$ we have used the fact that $\int_{\mathcal{L}^A(y)}p^A=1$ and $\int_{\mathcal{L}^B(z)}p^B=1$.

This is now the step where the Liouville hypothesis on the lamination plays a crucial role :

We have

$$\int_{\mathcal{L}^A}|\nabla_\xi p^A|\ dvol_{\mathcal{L}^A}\le C\left(\int_{\mathcal{L}^A}p^A(x,\xi,t)\right)^{\frac12}\left(\int_{\mathcal{L}^A}\frac{|\nabla_\xi p^A|^2}{p^A}\right)^{\frac12} \tag{III.17}$$

moreover a short computation shows that for $t>0$

$$\frac{\partial(p^A\ \log p^A)}{\partial t}+\Delta(p^A\ \log p^A)=-\frac{|\nabla p^A|^2}{p^A} \tag{III.18}$$

Thus

$$\int_{\mathcal{L}^A}\frac{|\nabla_\xi p^A|^2}{p^A}=-\frac{d}{dt}\int_{\mathcal{L}^A}p^A\ \log p^A \tag{III.19}$$

Combining (III.17) and (III.19) we obtain

$$\int_{\mathcal{L}^A} |\nabla_\xi p^A| \leq (1-\delta)^2 \int_{\mathcal{L}^A} p^A - \frac{C}{(1-\delta)^2} \frac{d}{dt} \int_{\mathcal{L}^A} p^A \log p^A \tag{III.20}$$

Integrating on time and on $\mathcal{S}^A_\varepsilon$ we get

$$\begin{aligned}&\int_{\mathcal{S}^A_\varepsilon} \frac{1}{T} \int_0^T \frac{1}{1-\delta} \int_{\mathcal{L}^A} |\nabla_\xi p^A| \\ &\leq (1-\delta)|\mathcal{S}^A_\varepsilon| - \frac{C}{(1-\delta)^3} \frac{1}{T} \int_{\mathcal{S}^A_\varepsilon} \int_{\mathcal{S}^A_\varepsilon} p^A(x, d\xi, t) \log p^A\end{aligned} \tag{III.21}$$

and using the Liouville property expressed in term of the cancellation of the Kaimanovich Entropy we have

$$\lim_{T\to+\infty} \frac{1}{T} \int_{\mathcal{S}^A_\varepsilon} \int_{\mathcal{S}^A_\varepsilon} p^A(x, d\xi, T) \log p^A(x, \xi, T) = 0$$

Choosing now δ_T so that we have

$$\lim_{T\to+\infty} \frac{1}{(1-\delta_T)^3} \frac{1}{T} \int_{\mathcal{S}^A_\varepsilon} \int_{\mathcal{S}^A_\varepsilon} p^A(x, d\xi, T) \log p^A(x, \xi, T) = 0 \tag{III.22}$$

and

$$\lim_{T\to+\infty} \frac{1}{(1-\delta_T)^3} \frac{1}{T} \int_{\mathcal{S}^B_\varepsilon} \int_{\mathcal{S}^B_\varepsilon} p^B(x, d\xi, T) \log p^B(x, \xi, T) = 0 \tag{III.23}$$

combining (III.16) and (III.21) we obtain

$$\begin{aligned}&\lim_{T\to+\infty} \int_{\mathcal{S}^{A,A,B}_\varepsilon} \frac{1}{T^3} \int_{[0,T]^3} \sum_{k,l,m} \delta^{k+l+m} |\partial \mathcal{L}^A_{k,\delta}(x,t)| \, |\mathcal{L}^A_{l,\delta}(y,s)| \, |\mathcal{L}^B_{m,\delta}(z,\sigma)| \\ &= 0\end{aligned} \tag{III.24}$$

So extending easily this result to each term of the right-hand side of (III.15) we deduce, using (III.8) and (III.14), that, for a.e. (x,y,z) in $\mathcal{S}^{A,A,B}_\varepsilon$, we have

$$\begin{aligned}&\lim_{T\to+\infty} \frac{1}{T^3} \int_{[0,T]^3} \sum_{k,l,m} \delta_T^{k+l+m} \operatorname{rlk}\left(\mathcal{D}^A_{k,\delta_T}(x,t) | \mathcal{D}^A_{l,\delta_T}(y,s), \mathcal{D}^B_{m,\delta_T}(z,\tau)\right) \\ &= \Lambda^\star(x,y,z)\end{aligned} \tag{III.25}$$

Integrating this identity on $\mathcal{S}^{A,A,B}_\varepsilon$ and making ε tend to 0 we get the desired result and theorem I.1 is proved.

IV Proofs of theorems I.2 and I.3.

IV.1 Proof of theorem I.2.

We use the same notations as in the previous section and the outline of the proof will look very much the same.

Take $\mathcal{S}_\varepsilon^{A_i}$ to be a saturated subset of the lamination defined by dA_i in $\mathcal{T}_i$ such that

- $|dA_i| \geq c > 0$ on $\mathcal{S}_\varepsilon^{A_i}$
- $\left|\int_{D_i} dA_i - \mathrm{Flux} dA_i\right| \leq \varepsilon$ where D_i is a given section of $\mathcal{T}_i$.

The existence of such a set is a simple consequence of the lamination hypothesis. We use the following notation $\mathcal{S}_\varepsilon = \mathcal{S}_\varepsilon^{A_1} \times \mathcal{S}_\varepsilon^{A_2} \times \mathcal{S}_\varepsilon^{A_3}$. We clearly have

$$\int_{\mathcal{S}_\varepsilon} \frac{|dA_1|(x_1)\ |dA_2|(x_2)\ |dA_3|(x_3)}{|x_1 - x_2|^3\ |x_1 - x_3|^3} \geq 16\pi^4 \int_{\mathcal{S}_\varepsilon} |g^{A_1A_2A_3}| \tag{IV.1}$$

Introduce p^{A_i} and $\tilde{p}^{A_i}$ like in section II. We have

$$\int_{\mathcal{S}_\varepsilon} |g^{A_1A_2A_3}| = \int_{\mathcal{S}_\varepsilon} \lim_{T\to+\infty} \frac{1}{T^3} \int_{[0,T]^3} \int_{\mathcal{S}_\varepsilon} \tilde{p}_{A_1}\ \tilde{p}_{A_2}\ \tilde{p}_{A_3}\ |g^{A_1A_2A_3}| \tag{IV.2}$$

Like for (III.10) we have

$$\begin{aligned} &\int_{\mathcal{S}_\varepsilon} \tilde{p}_{A_1}\ \tilde{p}_{A_2}\ \tilde{p}_{A_3}\ |g^{A_1A_2A_3}| \\ &= \sum_{k_1,k_2,k_3} \delta^{k_1+k_2+k_3} \int_{\prod_i \mathcal{L}^{A_i}_{k_i,\delta}(x_i,t_i)} |g^{A_1A_2A_3}| \bigwedge_i dvol_{\mathcal{L}^{A_i}(x_i)} \end{aligned} \tag{IV.3}$$

Recall at this step that $|\ |$ still denotes the norm for the canonical metric $g_{\mathbb{R}^4}$ on $\mathbb{R}^4$ but that $dvol_{\mathcal{L}^{A_i}(x_i)}$ is the volume form for the metric induced on the leaf by $g^{A_i} = g_{\mathbb{R}^4}/|dA_i|$. Arguing like for establishing (III.12) we get

$$|g^{A_1A_2A_3}| = |\iota_{\mathcal{L}^{A_1}\times\mathcal{L}^{A_2}\times\mathcal{L}^{A_3}}\mathcal{G}|_o \tag{IV.4}$$

where $\mathcal{G}$ is the Gauss form introduced in proposition II.1 and $|\ |_o$ is the scalar product for the metric on the leaves induced by g^{A_i}. Applying the definition of the Gauss-form we have

$$\iota_{\mathcal{L}^{A_1}\times\mathcal{L}^{A_2}\times\mathcal{L}^{A_3}}\mathcal{G} = V^* \left(\Omega(X) \wedge \Omega(Y)\right)$$

where $V\,:\ \mathcal{L}^{A_1}\times\mathcal{L}^{A_2}\times\mathcal{L}^{A_3}\longrightarrow S^3\times S^3$ given by (I.14) and Ω is the renormalized volume form on S^3 so that $\int_{S^3}\Omega=1$. We then have

$$\begin{aligned}&\int_{\prod_i \mathcal{L}^{A_i}_{k_i,\delta}(x_i,t_i)}|g^{A_1A_2A_3}|\ \bigwedge_i dvol_{\mathcal{L}^{A_i}(x_i)}\\&=\int_{\prod_i \mathcal{L}^{A_i}_{k_i,\delta}(x_i,t_i)}|V^*\left(\Omega\wedge\Omega\right)|_o\bigwedge_i dvol_{\mathcal{L}^{A_i}(x_i)}\end{aligned}\tag{IV.5}$$

Arguing exactly like in the previous section we can, here also, replace the $\mathcal{L}^{A_i}_{k_i,\delta}$ by the $\mathcal{D}^{A_i}_{k_i,\delta}$. Indeed this require the adding of small caps that we can choose to be contained respectively in $\mathcal{T}_1$, $\mathcal{T}_2$ and $\mathcal{T}_3$. Since the supports of the three laminations are disjoint we can ensure that $|V^*(\Omega\wedge\Omega)|_o$ is uniformly bounded for a triple of points in $\mathcal{T}_1\times\mathcal{T}_2\times\mathcal{T}_3$ where we choosed g^{A_i} to be the standard metric $g_{\mathbb{R}^4}$ on each additional cap $C_{\partial\mathcal{L}^{A_i}_{k_i,\delta}(x_i,\delta)}$ (it has no importance as long as the chosen metric is bounded from above and bellow relative to the standard one). Then we need to choose a good δ_T depending on T exactly like in the previous section in order to ensure

$$\lim_{T\to+\infty}\frac{1}{(1-\delta_T)^3}\frac{1}{T}\int_{\mathcal{S}^{A_i}_\varepsilon}\int_{\mathcal{S}^{A_i}_\varepsilon}p^{A_i}(x,d\xi,T)\,\log p^{A_i}(x,\xi,T)=0\tag{IV.6}$$

Like above the following choice of δ_T ensures

$$\begin{aligned}&\int_{\mathcal{S}_\varepsilon}\lim_{T\to+\infty}\frac{1}{T^3}\int_{[0,T]^3}\sum_{k_1,k_2,k_3}\delta^{k_1+k_2+k_3}\int_{\prod_i \mathcal{L}^{A_i}_{k_i,\delta}(x_i,t_i)}|V^*\left(\Omega\wedge\Omega\right)|_o\ \bigwedge_i dvol_{\mathcal{L}^{A_i}}\\&=\int_{\mathcal{S}_\varepsilon}\lim_{T\to+\infty}\frac{1}{T^3}\int_{[0,T]^3}\sum_{k_1,k_2,k_3}\delta^{k_1+k_2+k_3}\int_{\prod_i \mathcal{D}^{A_i}_{k_i,\delta}(x_i,t_i)}|V^*\left(\Omega\wedge\Omega\right)|_o\ \bigwedge_i dvol_{\mathcal{D}^{A_i}}\end{aligned}\tag{IV.7}$$

Using the Coarea formula of Federer, Since $|S^3|^2\,\Omega(X)\wedge\Omega(Y)$ is the standard volume form on $S^3\times S^3$ we have

$$\begin{aligned}&\int_{\prod_i \mathcal{D}^{A_i}_{k_i,\delta}(x_i,t_i)}|V^*\left(\Omega(X)\wedge\Omega(Y)\right)|_o\ \bigwedge_i dvol_{\mathcal{D}^{A_i}}\\&=\frac{1}{|S^3|^2}\int_{S^3\times S^3}d\sigma\ \text{Card}\left\{(x_i)_i\in\prod_i\mathcal{D}^{A_i}\ \text{s.t.}\ V((x_i)_i)=\sigma\right\}\end{aligned}\tag{IV.8}$$

For almost every choice of $\sigma=(u,v)\in S^3\times S^3$ the shadow of $\mathcal{D}^{A_2}_{k_2,\delta}(x_2,t_2)$ on $\mathcal{D}^{A_1}_{k_1,\delta}(x_1,t_1)$, relative to u, is a smooth curve and the projections of both this shadow and $\mathcal{D}^{A_3}_{k_3,\delta}(x_3,t_3)$ on a plane perpendicular to v are transverse and intersect each other along the set $\{(x_1,x_2,x_3)\in\prod_i\mathcal{D}^{A_i}$ s.t. $V(x_1,x_2,x_3)=\sigma\}$. To simplify the presentation we assume that Σ_1, Σ_2 and Σ_3 are connected. The definition of the asymptotic relative crossing number of Σ_1, Σ_2 and Σ_3 gives

$$\begin{aligned}&\mathrm{Card}\left\{(x_1,x_2,x_3)\in\prod_i\mathcal{D}^{A_i}\ \text{s.t. } V(x_1,x_2,x_3)=\sigma\right\}\geq\\ &\mathrm{arc}(\Sigma_1|\Sigma_2,\Sigma_3)\ \left|\prod_i\deg(\mathcal{D}^{A_i}_{k_i,\delta}(x_i,t_i);\Sigma_i)\right|\end{aligned}\tag{IV.9}$$

Combining (IV.3), (IV.5), (IV.7) and (IV.9) we then obtain

$$\begin{aligned}&\int_{\mathcal{S}_\varepsilon}\lim_{T\to+\infty}\frac{1}{T^3}\int_{[0,T]^3}\tilde{p}_{A_1}\ \tilde{p}_{A_2}\ \tilde{p}_{A_3}\ |g^{A_1A_2A_3}|\geq\mathrm{arc}(\Sigma_1|\Sigma_2,\Sigma_3)\times\\ &\int_{\mathcal{S}_\varepsilon}\bigwedge_i dx_i\lim_{T\to+\infty}\frac{1}{T^3}\int_{[0,T]^3}\sum_{k_1,k_2,k_3}\delta_T^{k_1+k_2+k_3}\left|\prod_i\deg(\mathcal{D}^{A_i}_{k_i,\delta}(x_i,t_i);\mathcal{T}_i)\right|\ \bigwedge_i dt_i\end{aligned}\tag{IV.10}$$

The degree of a $\mathcal{D}^{A_i}$ in $\mathcal{T}_i$ is, by definition, the intersection number of this surface with a section D_i of $\mathcal{T}_i$. So if ω_i is a 2-form in $\mathcal{T}_i$ Poincaré dual of D_i we have

$$\deg(\mathcal{D}^{A_i}_{k_i,\delta_T}(x_i,t_i);\mathcal{T}_i)=\int_{\mathcal{D}^{A_i}_{k_i,\delta_T}(x_i,t_i)}\omega_i$$

Let decompose $\int_{\mathcal{D}^{A_i}_{k_i,\delta_T}(x_i,t_i)}\omega_i=\int_{\mathcal{L}^{A_i}_{k_i,\delta_T}(x_i,t_i)}\omega_i+\int_{C_{\mathcal{L}^{A_i}_{k_i,\delta_T}(x_i,t_i)}}\omega_i$, since $|C_{\mathcal{L}^{A_i}_{k_i,\delta_T}(x_i,t_i)}|\leq c|\mathcal{L}^{A_i}_{k_i,\delta_T}(x_i,t_i)|$ we get, using (III.16), (III.21), (III.22) combined with (IV.10)

$$\begin{aligned}&\int_{\mathcal{S}_\varepsilon}\lim_{T\to+\infty}\frac{1}{T^3}\int_{[0,T]^3}\tilde{p}_{A_1}\ \tilde{p}_{A_2}\ \tilde{p}_{A_3}\ |g^{A_1A_2A_3}|\geq\mathrm{arc}(\Sigma_1|\Sigma_2,\Sigma_3)\times\\ &\int_{\mathcal{S}_\varepsilon}\bigwedge_i dx_i\lim_{T\to+\infty}\frac{1}{T^3}\int_{[0,T]^3}\prod_i\left|\int_{\mathcal{L}^{A_i}(x_i)}\tilde{p}^{A_i}(x_i,\xi_i,t_i)\omega_i\right|\end{aligned}\tag{IV.11}$$

It is not difficult to transpose the arguments of section II in order to replace $\tilde{p}^{A_i}$ by p^{A_i} in the previous inequality. We have $p^{A_i}(x_i,\xi_i,t_i)\ dvol_{\mathcal{L}^{A_i}}=$

$p^{A_i}(x_i, d\xi_i, t_i)$ and $\iota^*_{\mathcal{L}^{A_i}(x_i)}\omega_i = *\left(\omega_i \wedge \frac{dA_i}{|dA_i|}\right) dvol_{g_{\mathbb{R}^4}}$ where $*$ denotes the Hodge operator in $\mathbb{R}^4$. Using now the fact that on $\mathcal{L}^{A_i}$ $|dA_i|\ dvol_{g^{A_i}} = dvol_{g_{\mathbb{R}^4}}$ we finally obtain

$$\begin{aligned}&\int_{\mathcal{S}_\varepsilon} \lim_{T\to+\infty} \frac{1}{T^3} \int_{[0,T]^3} \tilde{p}_{A_1}\ \tilde{p}_{A_2}\ \tilde{p}_{A_3}\ |g^{A_1A_2A_3}| \geq \mathrm{arc}(\Sigma_1|\Sigma_2,\Sigma_3)\times \\ &\int_{\mathcal{S}_\varepsilon} \prod_i dx_i \lim_{T\to+\infty} \frac{1}{T^3} \int_{[0,T]^3} \prod_i \left| \int_{\pm^{A_i}_\varepsilon} p^{A_i} * (\omega_i \wedge dA_i) \right|\end{aligned} \tag{IV.12}$$

Using again the ergodic theorem II.1 for the right-hand-side of (IV.12) and combining this inequality with (IV.1) we have proved theorem I.2.

IV.2 Proof of theorem I.3.

The proof of this theorem is strongly related to the proof of theorem 4.1 in [8]. Let Σ'_1, Σ'_2 Σ'_3 be 3 surfaces respectively in $\mathcal{T}_1$, $\mathcal{T}_2$ and $\mathcal{T}_3$ such that $\deg(\Sigma'_i; \mathcal{T}_i) \neq 0$. Let u be a generic vector in S^3 such that the projections of Σ'_1 and Σ'_2 on a 3-plane perpendicular to u are transverse to each other. Let

$$S_u = \left\{ x \in \Sigma'_1 \quad \text{such that } \exists y \in \Sigma'_2 \quad u = \frac{x-y}{|x-y|} \right\}$$

From the definition of $\deg(\Sigma'_i; \mathcal{T}_i)$ we have, since $\mathcal{T}_i$ retracts on Σ_i,

$$\deg(\Sigma'_i; \mathcal{T}_i)\ [\Sigma_i] = [\Sigma'_i] \qquad \text{in } H_2(\mathcal{T}_i) \simeq H_2(\Sigma_i)$$

So that

$$[S_u] = \deg(\Sigma'_1; \mathcal{T}_1)\ \deg(\Sigma'_2; \mathcal{T}_2)\ \sigma_{1,2} \qquad \text{in } H_1(\mathcal{T}_1) \simeq H_1(\Sigma_1)$$

Consider now v, a generic vector in S^3, chosen so that the half cylinder $\mathcal{C} = \{S_u + tv \quad t \in \mathbb{R}_+\}$ is transverse to $\partial\mathcal{T}_3$. For t sufficiently large ($t \geq t_0$), $S_u + tv$ does not cross $\mathcal{T}_3$ anymore and we can immerse 2-disks in $\mathbb{R}^4 \setminus \mathcal{T}_3$ to close $\mathcal{C}$ and to make it as a union of immersed disks transverse to $\partial\mathcal{T}_3$. We denote by $\mathcal{D} = (\mathcal{D}_i)_{i\in I}$ this family of disks ($\mathcal{D}_i = p_i(D^2)$). Let m_i be the number of component of $p_i^{-1}(\mathcal{T}_3)$ that represent a non trivial element in $H_2(\mathcal{T}_3; \partial\mathcal{T}_3)$. Since $\deg(\Sigma'_i; \mathcal{T}_i)$ is the intersection with an oriented section of $\mathcal{T}_3$ generating $H_2(\mathcal{T}_3, \partial\mathcal{T}_3)$ we have

$$\mathrm{rc}(\Sigma'_1|\Sigma'_2, \Sigma'_3) \geq \sum_{i\in I} m_i |\deg(\Sigma'_3; \mathcal{T}_3)| \tag{IV.13}$$

We claim that

$$m = \sum_{i \in I} m_i \geq \|\sigma_{1,2,3}\| \, |\deg(\Sigma_1'; \mathcal{T}_1)| \;\; |\deg(\Sigma_2'; \mathcal{T}_2)| \tag{IV.14}$$

This inequality is proved exactly following the proof of lemma 4.2 in [8]. Since $\pi_1(\mathcal{T}_3; \partial\mathcal{T}_3) = 0$, we can, following lemma 4.3 in [8], homotope the p_i relative to the ∂D^2's to make it as a m_i clean extension of the components of S_u. It is performed without increasing the number of essential components of the inverse image of $\mathcal{T}_3$ by p_i. So that, if $\overline{p}_i$ is the m_i-clean extension, $p_i^{-1}(\mathcal{T}_3)$ is made of m_i disjoint disks in D^2 that represent non trivial elements of $H_2(\mathcal{T}_3; \partial\mathcal{T}_3)$. Let E_i be this union of disks in D^2 and $F_i = D^2 \setminus E_i$. Then

$$\sum_{i \in I} [p_i(F_i)] = \deg(\Sigma_1'; \mathcal{T}_1) \; \deg(\Sigma_2'; \mathcal{T}_2) \; \sigma_{1,2,3} \qquad \text{in } H_2(\mathbb{R}^4 \setminus \mathcal{T}_3; \partial\mathcal{T}_3 \cup \partial\mathcal{T}_1)$$

We have $m_i - 1 = |\chi(F_i)|$ then $\sum_{i \in I} m_i \geq \sum_{i \in I} |\chi(F_i)|$ Thus we get (IV.14) and theorem I.3 is proved.

V An example where rlk$(\Sigma_1|\Sigma_2, \Sigma_3) = 0$ and arc$(\Sigma_1|\Sigma_2, \Sigma_3) > 0$

We give an example bellow (see figure 2) where

$$\mathrm{rlk}(\Sigma_1|\Sigma_2, \Sigma_3) = 0 \qquad \text{and} \qquad \mathrm{arc}(\Sigma_1|\Sigma_2, \Sigma_3) > 0$$

A short description of the link : (We mainly rely on figure 2) We take Σ_1, Σ_2 and Σ_3 so that $\Sigma_1 \simeq T^2$, $\Sigma_2 \simeq T^2$ and $\Sigma_3 = \hat{\Sigma}_3 \cup \tilde{\Sigma}_3$ where $\hat{\Sigma}_3$ and $\tilde{\Sigma}_3$ are two disjoint torii. We slice $\mathbb{R}^4$ by hyperplanes H_t perpendicular to a fixed direction e for $-1 \leq t \leq 1$. Let $\Sigma = \cup \Sigma_i$, $H_t \cap \Sigma$ is singular at exactly 8 dates $-t_4 = -1 < -t_3 < -t_2 < -t_1 < 0 < t_1 < t_2 < t_3 < t_4 = +1$: Slicing increasingly in time we have

- for $t < -t_4 = -1$ $H_t \cap S = \emptyset$
- for $-1 < t < -t_3$ $H_t \cap S$ is made of 4 unlinked circles.
- at $t = -t_3$ $\tilde{\Sigma}_3$ splits into two components as shown in figure 2.
- at $t = -t_2$ both Σ_1 and Σ_2 split into two components as shown in figure 2.

- at $t=-t_1$ $\hat{\Sigma}_3$ split into 2 components as shown in figure 2.
- Σ is exactly symmetric relative to H_0 except that between $t=0$ and $t=t_1$ the left component of Σ_2 on figure 2 rotates exactly one time around the left component of $\hat{\Sigma}_3$ so that a rigid disk bounding this component of Σ_2 will intersect Σ_1 and $\tilde{S}_3$ along respectively the generator of $H_1(\Sigma_1)$ and the generator of $H_1(\tilde{\Sigma}_3)$ given by the left components of Σ_1 and $\tilde{\Sigma}_3$ on figure 2 (for $H_t \cap \Sigma$ $-t_1 < t < t_1$).

The class $\sigma_{1,2}$ in $H_1(\Sigma_1)$ defined in the last part of section I and obtained from the intersection of Σ_1 with any manifold bounding Σ_2 has for representative the left component on figure 2 of $\Sigma_1 \cap H_t$ for $-t_1 < t < t_1$. It is clear from the figure that there exists a disk bounding this component and intersecting Σ_3 (either $\hat{S}_3$ or $\tilde{S}_3$) at exactly two points with opposite intersection numbers so that

$$\mathrm{rlk}(\Sigma_1|\Sigma_2,\Sigma_3)=0$$

We prove now that $\mathrm{arc}(\Sigma_1|\Sigma_2,\Sigma_3)=2$. From the previous remark we have $\mathrm{arc}(\Sigma_1|\Sigma_2,\Sigma_3)\leq 2$. So we have to prove that $\mathrm{arc}(\Sigma_1|\Sigma_2,\Sigma_3)\geq 2$. If Σ_1', Σ_2' and $\Sigma_3'=\hat{\Sigma}_3'\cup\tilde{\Sigma}_3'$ are three surfaces in $\mathcal{T}_1$, $\mathcal{T}_2$ and $\mathcal{T}_3=\hat{\mathcal{T}}_3\cup\tilde{\mathcal{T}}_3$ with degrees $d_1=\deg(\Sigma_1';\mathcal{T}_1)$, $d_2=\deg(\Sigma_2';\mathcal{T}_2)$ and $(\tilde{d}_3,\hat{d}_3)=(\deg(\hat{\Sigma}_3';\hat{\mathcal{T}}_3),\deg(\tilde{\Sigma}_3';\tilde{\mathcal{T}}_3))$, then for obvious homological reasons (see the previous section) any generic 3-manifold bounding Σ_2' intersects Σ_1' along a 1-manifold homologous to $d_1\, d_2\, \sigma_{1,2}$ in $H_1(\mathcal{T}_1)\simeq H_1(\Sigma_1)$. Let Γ be one of the connected components of this manifold. Observe from figure 2 that there exists a class $[\tau]$ in $H_1(\Sigma_1)$ which admits a representative τ whose intersection number with $\sigma_{1,2}$ is $+1$, and which is bounded by a disk δ in $\mathbb{R}^4$ that does not intersect $\mathcal{T}_3$. Both $\sigma_{1,2}$ and τ generate $H_1(\mathcal{T}_1)$. If $\Gamma=d\sigma_{1,2}+\nu[\tau]$ and if Δ is any disk bounding Γ and intersecting Σ_3' transversally it suffices then to prove that

$$\frac{\mathrm{Card}(\Delta\cap\hat{\Sigma}_3')}{d\,\hat{d}_3}+\frac{\mathrm{Card}(\Delta\cap\tilde{\Sigma}_3')}{d\,\tilde{d}_3}\geq 2 \tag{V.15}$$

Without changing $\Delta\cap\mathcal{T}_3$ we can modify it in the following way : we make a small surgery by adding ν times the disk δ to Δ so that the boundary of the new disk obtained $\tilde{\Gamma}=\partial\tilde{\Delta}$ is homologous to $d\,\sigma_{1,2}$ we can then add to it a 2-dimensional annulus contained in $\mathcal{T}_1$ so that the final disk bounds $d\,\sigma$ where σ is any curve generating $\sigma_{1,2}$. Thus we replace Γ by d times the curve σ_0 given in figure 2 by the left component of $H_t\cap\Sigma_1$ and Δ is now an immersed disk f_*D^2 ($f : (D^2; \partial) \to (\mathbb{R}^4; \sigma_0)$) bounding $d\,\sigma_0$ and we can always assume that it intersects $\mathcal{T}_3$ transversally and that $f : \partial D^2 \to \sigma_0 \simeq S^1$ is monotonic. Let m be the number of essential components of $f^{-1}(\tilde{\mathcal{T}}_3)$: the components whose image by f are non 0 in $\pi_2(\tilde{\mathcal{T}}_3; \partial\tilde{\mathcal{T}}_3)$ (has a non zero intersection number with $\tilde{S}_3$). Arguing like in [8] (lemma 4.2 and 4.3) we can deform Δ without increasing it's number of intersection with Σ_3' such that every components of $f^{-1}(\mathcal{T}_3)$ is a disk whose image by f is essential. Let N be a 3-manifold in $\mathbb{R}^4 \setminus \hat{\mathcal{T}}_3$ bounding $\tilde{\Sigma}_3$ such that $N \cap H_t$ is made of embedded disks bounding the components of $\tilde{\Sigma}_3 \cap H_t$. It can be chosen so that σ_0 and N intersect transversally at two points p_1 and p_2 we can also choose N so that Δ and $N \setminus \mathcal{T}_3$ intersects along smooth oriented curves which are closed or starting from p_1 arriving at $\partial\tilde{\mathcal{T}}_3$ or p_2 or starting from $\partial\tilde{T}_3$ arriving at p_2. Observe that N is chosen such that N and $\mathcal{T}_3$ are separable (There exists an isotopy of $\mathbb{R}^4$ sending N and $\mathcal{T}_3$ into opposite sides of an hyperplane in $\mathbb{R}^4$). Observe also that since $\partial N = \tilde{S}_3$, the intersection number of $N \cap \partial\mathcal{T}_3$ with an oriented section of the tubular neighborhood $\tilde{T}_3 \simeq D^2 \times T^2$ is $+1$ and then the difference between the number of curves of $f^{-1}(N \setminus \tilde{\mathcal{T}}_3)$ coming from $f^{-1}(p_1)$ to a component of $f^{-1}(\tilde{\mathcal{T}}_3)$ and the number of curves of $f^{-1}(N \setminus \tilde{\mathcal{T}}_3)$ leaving this component to $f^{-1}(p_2)$ is equal to the intersection number of the image by f of this component with $\tilde{\Sigma}_3$. Since $\tilde{\mathcal{T}}_3$ deforms smoothly to $\tilde{\Sigma}_3$ in $\mathbb{R}^4 \setminus \hat{T}_3 \cup \mathcal{T}_1$ we may always choose N so that $\partial\tilde{\mathcal{T}}_3 \setminus N$ is diffeomorphic to $[0, 1] \times \tilde{S}_3 \simeq [0, 1] \times T^2$. Then any connected curve on $\partial\mathcal{T}_3 \setminus N$ whose two ends intersect $N \cap \partial\tilde{\mathcal{T}}_3$ with total intersection number being equal to zero can be smoothly deform in $\partial\tilde{\mathcal{T}}_3$, keeping it's ends fixed, to a curve in $N \cap \partial\tilde{\mathcal{T}}_3$. Then we deduce that we can homotope f in $\mathbb{R}^4 \setminus \hat{T}_3$, keeping $f(\partial D^2)$ fixed, keeping the number of essential components of $f^{-1}(\tilde{\mathcal{T}}_3)$ and their intersection number with $\tilde{S}_3$ fixed, such that the number of curves of $f^{-1}(N \setminus \tilde{\mathcal{T}}_3)$ intersecting a component of $f^{-1}(\tilde{\mathcal{T}}_3)$ is equal to the intersection number between this component with $\tilde{\Sigma}_3$ (at this stage we take into account the sign of the intersection number, so that if it is positive we only have curves arriving from $f^{-1}(p_1)$ and if it is negative curves leaving for $f^{-1}(p_2)$). Since f : $\partial D^2 \to \sigma$ is monotonic $f^{-1}(p_1)$ is made of exactly d points alternated with $f^{-1}(p_2)$ which is also made of exactly d points. Let $(\gamma_i)_{i\in I}$ to be the union of the connected curves among the one realizing $f^{-1}(N \setminus \tilde{\mathcal{T}}_3)$ that connect $f^{-1}(p_1)$ and $f^{-1}(p_2)$. Let C be a connected component of $D^2 \setminus \cup_i \gamma_i$. $f(\partial C)$ defines a class in $H_1(\mathbb{R}^4 \setminus \hat{\mathcal{T}}_3; \tilde{\mathcal{T}}_3 \cup N) \simeq H_1(\mathbb{R}^4 \setminus \hat{\mathcal{T}}_3) = \mathbb{Z}$ since $\tilde{\mathcal{T}}_3 \cup N$ is contractible to a point in $\mathbb{R}^4 \setminus \hat{\mathcal{T}}_3$. The intersection number of C with $\hat{S}_3$ gives the class in $\mathbb{Z}$. Let n be this number. Let q be the algebraic number of oriented arcs in $\partial D^2 \cap \partial C$ joining a point of $f^{-1}(p_1)$ and a point of $f^{-1}(p_2)$: the arc oriented by ∂D^2 is counted positively if it goes from a point of $f^{-1}(p_1)$ to a point of $f^{-1}(p_2)$ and negatively in the opposite case; in the first case it counts as $+1$ as a contribution to $H_1(\mathbb{R}^4 \setminus \hat{\mathcal{T}}_3; \tilde{\mathcal{T}}_3 \cup N)$ in the other case it counts as -1. The difference between q and the absolute number of arcs in $\partial D^2 \cap \partial C$ joining points of $f^{-1}(p_1)$ and $f^{-1}(p_2)$ is given by the number of arc of $f^{-1}(N \setminus \tilde{\mathcal{T}}_3)$ in C joining points of $f^{-1}(\{p_1\} \cup \{p_2\})$ and components of $f^{-1}(\tilde{\mathcal{T}}_3)$. Collecting all the informations above we easily get (V.15).

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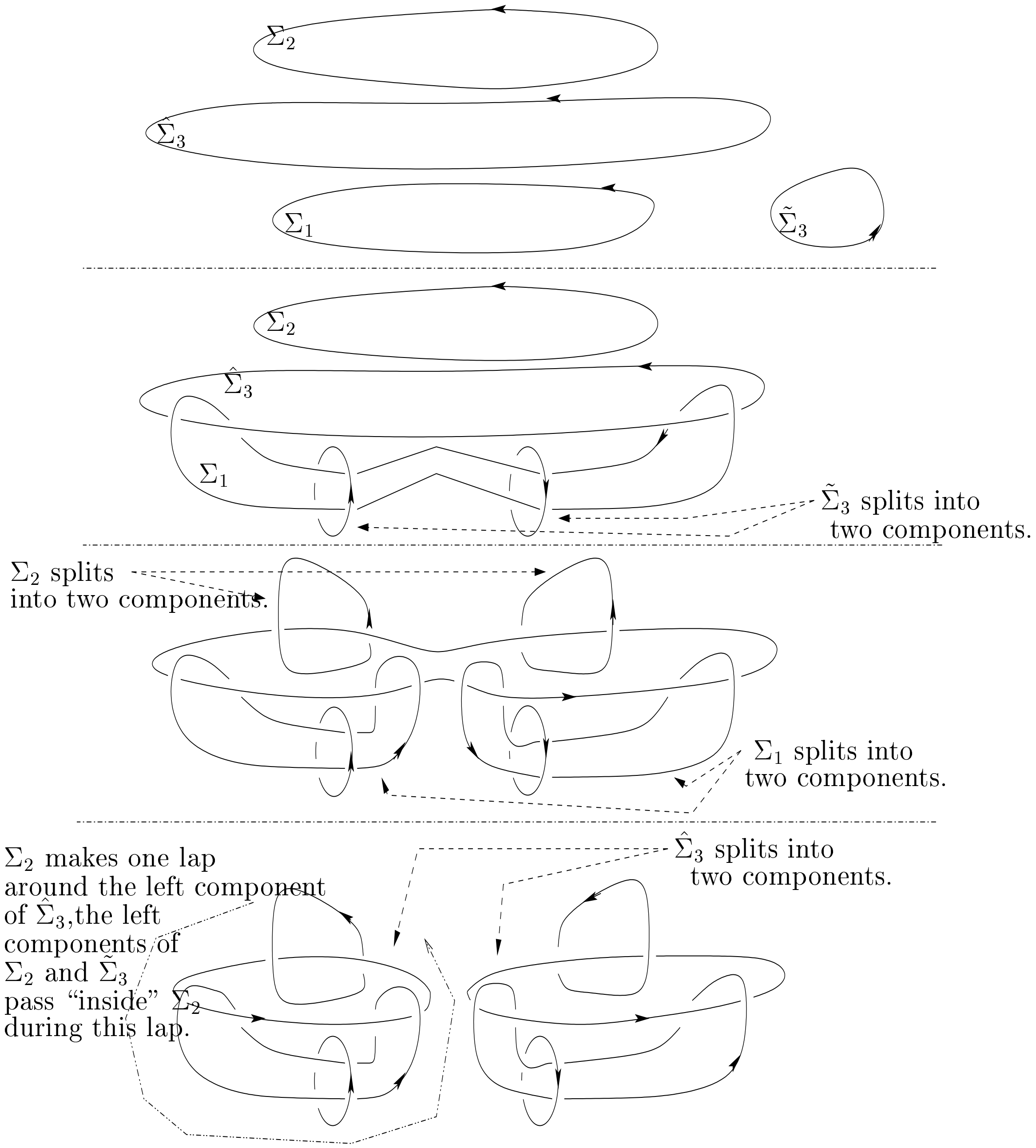

Figure 2: An example where $\mathrm{rlk}(\Sigma_1|\Sigma_2,\Sigma_3) = 0$ and $\mathrm{arc}(\Sigma_1|\Sigma_2,\Sigma_3) > 0$