WEAK CLOSURE OF SINGULAR ABELIAN L^p-BUNDLES IN 3 DIMENSIONS

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Abstract. We prove the closure for the sequential weak L^p-topology of the class of vectorfields on B^3 having integer flux through almost every sphere. We show how this problem is connected to the study of the minimization problem for the Yang-Mills functional in dimension higher than critical, in the abelian case.

1. Introduction

In this work we consider the class $L^p_{\mathbb{Z}}(B^3,\mathbb{R}^3)$ of vectorfields $X \in L^p(B^3,\mathbb{R}^3)$ such that

$$\int_{\partial B^3_r(a)} X\cdot\nu \in \mathbb{Z},\quad \forall a\in B^3,\ \text{a.e. } r<\mathrm{dist}(a,\partial B^3),$$

where $\nu:\partial B^3_r(a)\to S^2$ is the outward unit normal vector.

We observe that for $p\geq 3/2$ this class reduces to the divergence-free vectorfields, and therefore we reduce o the "interesting" case $p\in[1,3/2[$. It is clear that this class of vectorfields is closed by strong L^p-convergence (see Lemma 2.5). We are interested in the closedness properties of $L^p_{\mathbb{Z}}(B^3,\mathbb{R}^3)$ for the sequential weak-L^p topology, and our main result in the present work is the following:

Theorem 1.1. *For $1<p<3/2$ the class $L^p_{\mathbb{Z}}(B^3,\mathbb{R}^3)$ is weakly sequentially closed. More precisely, whenever*

$$X_k\in L^p_{\mathbb{Z}}(B^3,\mathbb{R}^3),\quad X_k \overset{weak\text{-}L^p}{\rightharpoonup} X_\infty,$$

then $X_\infty\in L^p_{\mathbb{Z}}(B^3,\mathbb{R}^3)$.

For $p=1$ given any vector-valued Radon measure $X\in\mathcal{M}^3(B^3)$ where

$$\mathcal{M}^3(B^3):=\{(\mu_1,\mu_2,\mu_3)|\ \mu_i \text{ signed Radon measure on } B^3\},$$

we can find a sequence $X_k\in L^1_{\mathbb{Z}}(B^3,\mathbb{R}^3)$ such that $X_k\rightharpoonup X$ weakly in the sense of measures.

2. Motivation: Yang-Mills theory in supercritical dimension

In this section we show how theorem 1.1 can be used in the framework of Yang-Mills theory. Consider a principal G-bundle $\pi : P \to M$ over a compact Riemannian manifold M, and call $\mathcal{A}(P)$ the space of smooth connections on P. The celebrated *Yang-Mills functional* $YM : \mathcal{A} \to \mathbb{R}^+ \cup \{\infty\}$ is then defined as the L^2-energy of the curvature F_A of A:

$$YM(A) := \int_M |F_A|^2 d\,Vol_g.$$

Critical points of this functional are connections satisfying in a weak sense the Yang-Mills equations, which have been extensively studied in the conformal dimension 4, due to the geometric invariants that they help define (see [DK90, FU91]). We will focus here on the study of the functional YM from a variational viewpoint. We will just consider the two simplest and most celebrated cases $G = SU(2)$ (nonabelian case) and $G = U(1)$ (abelian case).

2.0.1. *The smooth class.* We first observe that since the Yang-Mills equations could have singularities, it is not clear if the infimum

$$\inf_{C^\infty_\varphi \ni A} YM(A)$$

(the subscript "φ" means that we are fixing a smooth boundary datum φ) is attained. The "natural" way to extend the class where YM is defined, would then be to allow more general L^2-curvatures. Since $F_A = dA + A \wedge A$, we would have to consider therefore $W^{1,2}$-regular connections, and $W^{2,2}$-change of gauge functions (for a detailed description of the theory of Sobolev principal bundles see for example [Weh04, Kes08, Iso09]).

Remark 2.1. *The fact that by the usual Sobolev embedding theorem $W^{2,2} \hookrightarrow C^0$ only when the dimension of the domain is < 4 implies that the topology of the bundles which we consider is fixed just in low dimension. We therefore call $n = 4$ the* critical *dimension in the study of the functional YM.*

2.1. **A parallel between the study of harmonic maps $u : B^3 \to S^2$ and that of YM in dimension 5.** The most celebrated problem in which the study of singularities in a variational setting was introduced, is the minimization of the Dirichlet energy

$$E(u) := \int_{B^3} |\nabla u|^2 dx \quad \text{for } u : B^3 \to S^2 \text{ with } u|_{\partial B^3} = \varphi.$$

As above, the infimum

$$\inf_{C^\infty_\varphi(B^3,S^2)} E$$

is in general *not* achieved, and in this case the natural space to look at would be the space of functions having one weak derivative in L^2, namely $W^{1,2}(B^3,S^2)$. We observe that for the functional E the critical dimension would then be 2, since $W^{1,2}(B^n,S^2)\hookrightarrow C^0$ just for $n<2$. The optimal result achieved in this case is Theorem 2.2 below. Such a regularity result would be the main goal in the study of the functional YM in dimension higher than critical (but we will see that extra difficulties arise when we deal with singular curvatures).

Theorem 2.2 ([SU82]). *Given a smooth map $\varphi\in C^\infty(S^2,S^2)$, for any minimizer u of E in $W^{1,2}_\varphi(B^3,S^2)$ there exist finitely many points $a_1,\ldots,a_N\in B^3$ and numbers $d_1,\ldots,d_N=\pm1$ such that*

$$\begin{gathered} u\in C^\infty(B^3\setminus\{a_1,\ldots,a_N\},S^2),\\ \deg(u,a_i)=d_i \textit{ for all } i. \end{gathered}$$

The singularities around which u realizes a nonzero integer degree as above [1] are called *topological singularities*.

If we want to consider an analogous topological obstructions for connections on bundles, we must start from the celebrated *Chern-Weil theory* [Zha01], which describes topological invariants of bundles via characteristic classes represented in terms of curvatures of connections. The most prominent "topological singularity" notion arising in relation to Yang-Mills $SU(2)$-gauge theory in dimension 4 is encoded into the second Chern class of the associated bundle. This homology class can be represented using the curvature of a smooth connection A via the Chern-Weil formula

$$c_2(P)=\left[-\frac{1}{8\pi^2}\mathrm{tr}(F_A\wedge F_A)\right].$$

In [Uhl85] it was proved that (in dimension 4) under a boundedness condition on the L^2 norm of the curvature, we have $c_2(P)\in\mathbb{Z}$. Such integrality condition has a role which is analogous to the one played by the integrality of the degree of maps $g:S^2\to S^2$ in the study of harmonic maps in $W^{1,2}(B^3,S^2)$, and as such is useful to study the YM functional in dimension 5. More precisely, the strategy [KR08, Kes08] consists in introducing the analogous of the space of maps with

[1] The realization of a degree around a point is a local topological obstruction for the strong approximability in $W^{1,2}$-norm [Bet90, BCDH91]. Global obstructions also play a role in approximability properties [HL03].

topological singularities described in Theorem 2.2. This time one has to consider smooth bundles defined on the base manifold with some finite set of points removed

$$\mathcal{P} := \left\{ \begin{array}{c} \text{principal } SU(2)\text{-bundles of the form} \\ P \to M \setminus \Sigma, \text{ for some finite set } \Sigma \subset M \end{array} \right\},$$

and then take the curvaturs of smooth connections on these bundles, realizing integral Chern numbers on small spheres surrounding the singularities:

$$\mathcal{R}^\infty := \left\{ F_A \,\middle|\, \begin{array}{l} A \text{ is a smooth connection on some } P \in \mathcal{P}, \\ c_2(P|_{\partial B(x,\varepsilon)}) \in \mathbb{Z} \setminus \{0\} \text{ for } x \in \Sigma, \forall \varepsilon < \operatorname{dist}(x, \partial B \cup \Sigma) \end{array} \right\}.$$

Motivated by the above analogy, we can say that *the above class $\mathcal{R}^\infty$ should be contained in any candidate for a class of critical points of YM in dimension 5*, as already noted in [Kes08, KR08]. If instead of considering the whole $\mathcal{R}^\infty$, we fix the number, degree and position of the singularities, a gap phenomenon arises, in analogy to [BCL86], as described in [Iso98, Iso08].

2.2. **The basic difficulty: singularities of bundles.** What is it that forbids to continue the analogy with harmonic maps, and to prove a regularity result like Theorem 2.2 for YM in dimension 5? To answer this question, we recall the two main ingredients without which such a result is not possible:

(A) A good *variational setting*, i.e. the presence of a class which contains the maps with topological singularities and in which a minimizing sequence of E has a converging subsequence. It is shown in [Bet91, Bet90] that for harmonic maps, this class is $W^{1,2}$ with the sequaential weak topology, indeed

$$C^\infty_\phi(B^3, S^2) \textit{ is weakly sequentially dense in } W^{1,2}(B^3, S^2),$$

and the wanted compactness property is a consequence of the Banach-Alaoglu theorem. In $W^{1,2}(B^3, S^2)$ therefore, the existence of a minimizer for E with fixed boundary datum is clear, and therefore the *existence of weak solutions* for the equation of critical points of E is established.

(B) An *ε-regularity theorem*, i.e. the implication

$$E(u) \le \varepsilon \textit{ on } B_1 \textit{ and } u \textit{ is a minimizer of } E \\ \Rightarrow \\ u \textit{ is Hölder on } B_{1/2}.$$

This kind of result is used to prove the discreteness of the singularity set, which is the main difficulty in the proof of Theorem 2.2.

We see that already the ingredient (A) (which is needed in order to formulate (B)) is problematic in the case of the YM functional on bundles, since nothing shows that *a priori* the singularities of a minimizer should not accumlate, and it is not clear how to define a bundle with *accumulating topological singularities*, at least if we stick to the differential-geometric definition of a bundle. We can formulate the following general problem:

Open Problem 1 (Right variational setting for the Yang-Mills theory in dimension 5)**.** *Find a topological space*

(1) *which includes the class $\mathcal{R}^\infty$ of curvatures with finitely many singularities*
(2) *in which minimizing sequences for YM are compact.*

The natural candidate for a space solving the above problem in the nonabelian case is the space of L^2*-curvatures on singular* $SU(2)$*-bundles* defined in [KR08], Definition III.2. Such class clearly contains $\mathcal{R}^\infty$; Open Problem 1 is then equivalent to asking to find a suitable topology on this natural class such that minimizing sequences for YM are compact. Not only is the above problem still open, but *it is not known whether for all smooth boundary data the infimum of YM is achieved, even in the class of L^2-curvatures of* [KR08].

2.3. **Main Result.** In this work we obtain the good setting described above (thereby solving Open Problem 1), in the simpler case of *abelian bundles*, i.e. bundles with gauge group $U(1)$.

Smooth principal $U(1)$-bundles on a 2-manifold Σ correspond to hermitian complex line bundles, and are classified by the integer given (see [MS74] for example) by the first Chern class, again expressable via the Chern-Weil theory by

$$c_1(P) := \frac{1}{2\pi}\int_\Sigma F_A.$$

We are lead by the analogy with the above discussion about the nonabelian case in dimension 5, to consider the YM functional in dimension 3, where the point singularities would be classified by the value of

$$[c_1(P|_{\partial B_\varepsilon(x)})] \in H^2(B^3, \mathbb{Z})$$

for any sufficiently small sphere $\partial B_\varepsilon(x)$ near an isolated singular point x. The corresponding classes $\mathcal{P}$ and $\mathcal{R}^\infty$ for $U(1)$-bundles, are defined as above, substituting "$SU(2), c_2(P|_{\partial B^5(x)})$" with "$U(1), c_1(P|_{\partial B^3(x)})$".

Remark 2.3. *We are helped in our approach by the fact that in the abelian case the curvature of a $U(1)$-bundle over B^3 projects to a well-defined curvature 2-form on B^3 (see for example* [KN96a, KN96b]*), thereby simplifying our definition. This is the reason why our results do not immediately generalize to the nonabelian case.*
Up to a universal constant, we can suppose that the projected 2-forms have integral in $\mathbb{Z}$ along any closed surface (possibly containing singular points).

We have therefore the following candidate for the class seeked in Open Problem 1:

Definition 2.4 (L^p-curvatures of singular $U(1)$-bundles [KR08], Definition II.1)**.** *An L^p-curvature of a singular $U(1)$-bundle over B^3 is a measurable real-valued 2-form F satisfying*

- $$\int_{B^3} |F|^p dx^3 < \infty,$$
- *For all $x \in B^3$ and for almost all $0 < r < \operatorname{dist}(x, \partial B^3)$ we have*
$$\int_{\partial B_r(x)} i^*_{\partial B_r(x)} F \in \mathbb{Z},$$
where $i^*_{\partial B_r(x)}$ is the inclusion map of $\partial B_r(x)$ in B^3.

We call $\mathcal{F}^p_{\mathbb{Z}}(B^3)$ the class of all such 2-forms F.

We observe that the above class is clearly closed in the *strong* L^p-topology:

Lemma 2.5. *The class $\mathcal{F}^p_{\mathbb{Z}}(B^3)$ is closed for the L^p topology.*

Proof. We take a sequence $F_k \in \mathcal{F}^p_{\mathbb{Z}}(B^3)$ such that $F_k \stackrel{L^p}{\to} F_\infty$. If we take $x \in B^3, R < \operatorname{dist}(x, \partial B^3)$, then there holds

$$||F_k - F_\infty||^p_{L^p} \geq \int_{B_R(x)} |F_k - F_\infty|^p dx \geq \int_0^R \left| \int_{\partial B_r(x)} i^*_{\partial B_r(x)}(F_k - F_\infty) d\mathcal{H}^2 \right|^p dr.$$

Therefore the above L^p-functions

$$f_k : [0, R] \to \mathbb{Z},\ f_k(r) := \int_{\partial B_r(x)} i^*_{\partial B_r(x)} F_k d\mathcal{H}^2$$

converge to the analogously defined function f_∞ in L^p, therefore also pointwise almost everywhere, thus proving that F_∞ also satisfies the properties in Definition 2.4. $\square$

Remark 2.6. *It has been already proved in* [Kes08, KR08] *that the class $\mathcal{R}^\infty$ is dense in $F^p_{\mathbb{Z}}$ for the L^p-topology (see also* [Pet]*).*

The fact that $c_1(P|_{\partial B_\varepsilon(x)}) \neq 0$ for all ε small enough implies that the curvature is not in L^p for $p \geq 3/2$ (for example the form $F = (4\pi r^2)^{-1} d\theta \wedge d\phi \in \Omega^2(B^3 \setminus \{0\})$ represents the curvature of an $U(1)$-bundle over $B^3 \setminus \{0\}$ having $c_1 = 1$ on all spheres containing the origin, and it is an easy computation to show that $F \notin L^{3/2}$. Therefore the study of the above defined YM functional (i.e. the one equal to the L^2-norm of the curvature) is trivial, no topological charge being possible. Therefore we study a similar functional where we substitute the L^p-norm to the L^2-norm:

$$YM_p(P) = \int_M |F_A|^p dx^3.$$

From the above discussion it follows that singularities realizing nontrivial first Chern numbers arise only if $p < 3/2$. For such p the class $\mathcal{F}^p_{\mathbb{Z}}(B^3)$ is in bijection with the class $L^p_{\mathbb{Z}}(B^3, \mathbb{R}^3)$ present in Theorem 1.1, via the identification of k-covectors β with $(n-k)$-vectors $*\beta$ in $\mathbb{R}^n$, given by imposing

$$\langle \alpha, *\beta \rangle = \langle \alpha \wedge \beta, \vec{e} \rangle \tag{2.1}$$

for all $(n-k)$-covectors α, where $\vec{e}$ is an orientating vectorfield of $\mathbb{R}^n$. After this identification, we can reformulate Theorem 1.1 as follows:

Theorem 2.7 (Main Theorem)**.** *If $p > 1$ and $F_n \in \mathcal{F}^p_{\mathbb{Z}}(B^3)$ and $||F_n||_{L^p} \leq C < \infty$ then we can find a subsequence $F_{n'}$ converging weakly in L^p to a 2-form in $\mathcal{F}^p_{\mathbb{Z}}(B^3)$.*

This answers Open Problem 1 in the case of $U(1)$-bundles:

Corollary 2.8 (Solution to Open Problem 1 in the case of $U(1)$-bundles)**.** *In the case of $U(1)$-bundles, the class $\mathcal{F}^p_{\mathbb{Z}}(B^3)$ with the sequential weak L^p-topology solves Open Problem 1 when $1 < p < 3/2$.*

A direct consequence of the above two results is the existence of minimizing $U(1)$-curvatures in $\mathcal{F}^p_{\mathbb{Z}}(B^3)$ under extra constraints, such as for example the imposition of a nontrivial boundary datum[2].

[2] We observe that defining the Dirichlet boundary value minimization problem for YM_p on $\mathcal{F}^p_{\mathbb{Z}}(B^3)$ is a delicate issue, which will therefore be treated separately (see [Pet]).

2.4. **Main points of the proof and outline of the paper.** The counterexample in Proposition 8.1 shows that Theorem 2.7 cannot hold in case $p=1$, as in such case we obtain all currents and we possibly loose any integrality condition by weak convergence. This means that the convexity of the L^p-norm arising when $p>1$ is really needed for a result similar to Theorem 2.7 to hold. On the other hand, the notions of a minimal connection as in [BCL86] or in [Iso98] are based on the duality between currents and smooth functions, where again no convexity is involved. Therefore we face the difficulty of finding a strategy more adapted to our problem. A difficulty arising from the presence of a L^p-exponent different from 1 arised also in the work of Hardt and Rivière [HR03], where an extension of the the Cartesian Currents notion of a minimal connection had to be introduced in order to treat singularities of functions in $W^{1,3}(B^4,S^2)$. For such definition one had to consider the class of *scans*, which are roughly generalized currents where the mass of slices had to be taken in L^α-norm with $\alpha<1$ (instead of $\alpha=1$, which would give back the usual mass as in [Fed69], 4.3). In order to achieve the weak compactness result analogous to our Theorem 2.7, a particular distance between scans was introduced, which allowed a $L^{1/\alpha,\infty}$-estimate. Such procedure was inspired by the approach of Ambrosio and Kirchheim [AK00], Sections 7 and 8, which used BV (instead of $L^{1/\alpha,\infty}$) bounds for functions with values in a suitable metric space, obtaining a rectifiability criterion and a new proof of the closure theorem for integral currents, via a maximal function estimate.

In the case of [AK00] the metric space considered was the one of rectifiable currents arising as slices of an initial current, with the flat metric. In [HR03] a distance d_e extending the definition of the flat metric was considered on the space of scans arising as slices of graphs. In our case we introduce a metric on the space Y of L^p-forms arising as slices on concentric spheres of a given curvature $F\in\mathcal{F}^p_{\mathbb{Z}}$:

$$Y:=L^p(S^2)\cap\left\{h:\int_{S^2}h\in\mathbb{Z}\right\}.$$

Our definition will be the following:

$$d(h_1,h_2):=\inf\left\{\|X\|_{L^p}:\ h_1-h_2=\mathrm{div}X+\partial I+\sum_{i=1}^N d_i\,\delta_{a_i}\right\},$$

where the infimum is taken over all triples given by a L^p-vectorfield X, an integer 1-current I of finite mass, and a finite set of integer degree singularities, given by an N-ple of couples (a_i,d_i), where $a_i\in S^2$ and

$d_i \in \mathbb{Z}$.

The fact that d is a metric is not immediate (see *Section 3*): in particular the implication

$$d(h_1, h_2) = 0 \Rightarrow h_1 = h_2$$

depends upon a result (see [Pet10], and Proposition 3.5) which says that flow lines of a L^p-vectorfield on S^2 with $\mathrm{div} V = \partial I$ where I is an integer multiplicity rectifiable 1-current of finite mass can be represented as preimages $u^{-1}(y)$, $y \in \mathbb{S}^1$, for some $u \in W^{1,p}(S^2, S^1)$.

The estimate connecting the distance d above to the ideas of [AK00, HR03] is (see Proposition 4.1) a bound on the lipschitz constant of the slice function

$$h : [s', s] \to (Y, d), \quad x \mapsto h(x) := T_x^* i^*_{\partial B_x(a)} F,$$

where $T_x(\theta) := a + x\theta$ maps $\partial B_x(a)$ to S^2. We estimate the lipschitz constant of h in terms of the maximal function of the L^1-function

$$f : [s', s] \to \mathbb{R}^+, \quad f(x) := \|h(x)\|^p_{L^p},$$

an estimate in the same spirit of the one used in [HR03], which was a generalization of the pioneering approach of [AK00].

In *Section 4* we prove a modified version of Theorem 9.1 of [HR03], which from the uniform $L^{p,\infty}$-bound on a sequence of maximal functions Mf_n defined as above (which is a direct consequence of the uniform L^p-bound on the sequence of curvatures F_n considered initially), allows us to deduce a kind of *locally uniform* pointwise convergence of the slices $h_n(x)$ for a.e. x, up to the extraction of a subsequence. This uniformity is the main advantage of our whole construction, and this is why we have to introduce the above distance and maximal estimate. The seed from which our technique grew was planted by [AK00], and first developed in [HR03].

Section 6 is devoted to the verification of the hypotheses of the abstract Theorem 5.1, and *Section 7* concludes that we can extract a subsequence as requested by Theorem 2.7, which easily follows from Theorem 5.1.

The last *Section 8* is devoted to the justification of the hypotheses on the exponent p in theorem 2.7.

3. Definition of the metric

We consider the following function on 2-forms[3] in $L^p(\wedge^2 D)$, for some smooth domain $D \subset \mathbb{R}^2$ or for $D = S^2$, and for $1 < p < 3/2$:

$$d(h_1, h_2) = \inf\left\{ ||X||_{L^p} :\ h_1 - h_2 = \mathrm{div}X + \partial I + \sum_{i=1}^{N} d_i \delta_{a_i} \right\},$$

where the above infimum is taken over all triples given by a L^p-vectorfield X, an integer 1-current I of finite mass, and a finite set of integer degree singularities, given by an N-ple of couples (a_i, d_i), where a_i are points in D and the numbers $d_i \in \mathbb{Z}$ represent the topological degrees of X near the singularities $\{a_i\}$.

Remark 3.1. *We observe that up to changing the current I and the singularity set in any above triple, we may reduce to considering the case where the singularity set contains only one given point, which we call 0. More precisely, we can say that an equivalent formulation of the above distance is*

$$d(h_1, h_2) = \inf\left\{ ||X||_{L^p} :\ h_1 - h_2 = \mathrm{div}X + \partial I + d\delta_0 \right\},$$

where the infimum is now taken on all triples (X, I, d), where X, I are as above, and d is an integer number.

Remark 3.2. *In particular, from the above we obtained that*

$$d(h_1, h_2) \neq \infty \text{ implies } \int_D (h_1 - h_2) \in \mathbb{Z}. \tag{3.1}$$

Therefore the following function $\tilde{d}$ is a priori different than d (and can be seen as an extension of d)

$$\tilde{d}(h_1, h_2) = \inf\left\{ ||X||_{L^p} :\ h_1 - h_2 = \mathrm{div}X + \partial I + \delta_0 \int_D (h_1 - h_2) \right\}, \tag{3.2}$$

since there is no apparent reason for it to be infinite in the case when $\int_D (h_1 - h_2) \notin \mathbb{Z}$.

Proposition 3.3. *The above defined function d is a metric on $L^p(\wedge^2 D)$, both in the case when $D = [0,1]^2$ and in the case $D = S^2$.*

Proof. We will prove the three characterizing properties of a metric.

[3]In order to avoid heavy notations, we will often use the formula (2.1), identifying k-vectors with $(n-k)$-differential forms in an n-dimensional domain, without explicit mention.

- *Reflexivity:* This is clear since both the L^p-norm, the space of integer 1-currents of finite mass and the space of finite sums $\sum_{i=1}^N d_i\delta_{a_i}$ as above, are invariant under sign change.
- *Transitivity:* If we can write

$$\begin{cases} h_1 - h_2 = \mathrm{div}X_\varepsilon + \partial I_\varepsilon + \sum_{i=1}^N d_i\delta_{a_i} \\ h_2 - h_3 = \mathrm{div}Y_\varepsilon + \partial J_\varepsilon + \sum_{j=1}^M e_j\delta_{b_j}, \end{cases}$$

where

$$\begin{cases} ||X_\varepsilon||_{L^p} \le d(h_1,h_2) + \varepsilon \\ ||Y_\varepsilon||_{L^p} \le d(h_2,h_3) + \varepsilon, \end{cases}$$

then putting $Z_\varepsilon := X_\varepsilon + Y_\varepsilon$, $K_\varepsilon = I_\varepsilon + J_\varepsilon$ and considering the union of the singularity sets $\{(c_k,f_k)\} := \{(a_i,d_i),(b_j,e_j) : i = 1,\dots,N,\ j = 1,\dots,M\}$, then we see that K_ε is still an integer 1-current of finite mass and that $h_1 - h_3 = \mathrm{div}Z_\varepsilon + \partial K_\varepsilon + \sum_k f_k\delta_{c_k}$. Then we have:

$$d(h_1,h_3) \le ||Z_\varepsilon||_{L^p} \le ||X_\varepsilon||_{L^p} + ||Y_\varepsilon||_{L^p} \le d(h_1,h_2) + d(h_2,h_3) + 2\varepsilon,$$

and as $\varepsilon \to 0$ we obtain the transitivity.
- *Nondegeneracy:* This is the statement of the following proposition.

□

Proposition 3.4. *Under the hypotheses above, $d(h_1,h_2) = 0$ implies $h_1 = h_2$ almost everywhere, for $1 < p < 2$.*

We may suppose without loss of generality that $\int_D (h_1 - h_2) \in \mathbb{Z}$. We start by taking a sequence of forms X_ε such that

$$\begin{cases} ||X_\varepsilon||_{L^p} \to 0 \\ h_1 - h_2 = \mathrm{div}X_\varepsilon + \partial I_\varepsilon + \delta_0 \int_D (h_1 - h_2) \end{cases}$$

We would be almost done, if we could control also the convergence of the 1-currents I_ε. To do so, we start by expressing the boundaries ∂I_ε in divergence form. Therefore, we consider the equations

$$\begin{cases} \Delta\psi = h_1 - h_2 + \delta_0 \int_D (h_2 - h_1) \\ \int_D \psi = 0 \end{cases} \tag{3.3}$$

(by classical results, this equation has a solution whose gradient is in L^q for all q such that $q < 2$ and $q \le p$) and

$$\begin{cases} \Delta\varphi_\varepsilon = \mathrm{div}X_\varepsilon \\ \int_D \varphi_\varepsilon = 0 \end{cases} \tag{3.4}$$

This second equation can be interpreted in terms of the Hodge decomposition of the 1-form associated to X_ε: indeed, for a L^p 1-form α we know by classical results that it can be Hodge-decomposed as

$$\begin{cases} \alpha = df + d^*\omega + h, \text{ where} \\ \int_D f = 0,\ \int_D *\omega = 0,\ \Delta h = 0, \text{ and} \\ ||df||_{L^p} + ||d^*\omega||_{L^p} + ||h||_{L^p} \leq C_p||\alpha||_{L^p}, \end{cases}$$

Therefore in equation (3.4) we can associate (via formula (2.1)) a 1-form α to X_ε and take φ_ε equal to the function f coming from the above decomposition. Then an easy verification shows that (3.4) is verified.

We have thus, that both (3.3) and (3.4) have a solution, and such solutions satisfy the following estimates:

$$\begin{cases} ||\nabla\varphi_\varepsilon||_{L^p} \leq c_p||X_\varepsilon||_{L^p} \to 0 \\ \nabla\psi \in W^{1,p} \subset L^p \text{ since } p^* = \frac{2p}{2-p} > p. \end{cases}$$

Then (supposing $p < 2$) we obtain

$$\begin{cases} \partial I_\varepsilon = \operatorname{div}(\nabla(\varphi_\varepsilon - \psi)) \\ ||\nabla(\varphi_\varepsilon - \psi)||_{L^p} \text{ is bounded} \end{cases} \tag{3.5}$$

Now we consider the vector field $\nabla(\varphi_\varepsilon - \psi) := V_\varepsilon \in L^p(D, \mathbb{R}^2)$.

Proposition 3.5 ([Pet10]). *Suppose that we have a function $V \in L^p(D, \mathbb{R}^2)$ with $p > 1$, for a domain $D \subset \mathbb{R}^2$ or for $D = S^2$, whose divergence can be represented by the boundary of an integer 1-current I on D, i.e. for all test functions $\gamma \in C_c^\infty(D, \mathbb{R})$ we have*

$$\int_D \nabla\gamma(x) \cdot V(x)dx = \langle I, \nabla\gamma\rangle. \tag{3.6}$$

Then there exists a $W^{1,p}$-function $u : D \to S^1 \simeq \mathbb{R}/2\pi\mathbb{Z}$ such that $\nabla^\perp u = V$.

By Lemma 3.6, we can then write, because of (3.5),

$$\begin{cases} \nabla^\perp u_\varepsilon = \nabla(\varphi_\varepsilon - \psi) \\ \partial I_\varepsilon = \operatorname{div}(\nabla(\varphi_\varepsilon - \psi)) \\ ||\nabla u_\varepsilon||_{L^p} \leq C||\nabla(\varphi_\varepsilon - \psi)||_{L^p} \leq C. \end{cases}$$

Then we have that a subsequence u_k of the u_ε converges weakly in $W^{1,p}(D, \mathbb{R}^2)$ to a limit u_0, and thus it converges in L^1_{loc}, proving that also $u_0 \in W^{1,p}(D, \mathbb{R}^2)$. We also have that the u_k converge to u_0 almost everywhere, and thus we obtain that the limit function u_0 also has almost everywhere values in S^1. Since we know now that $u_k \stackrel{L^1}{\to} u_0$

and also that $||u_k||_{L^\infty} \leq 1$, we obtain by interpolation that $u_k \overset{L^r}{\to} u_0$ for all $r < \infty$. Therefore (by choosing $r = \frac{q}{q-1}$ and by Young's inequality) we have also that

$$\nabla^\perp u_k \overset{L^1}{\to} \nabla^\perp u_0 \tag{3.7}$$

By a generalization of Sard's theorem, the fibers $F_\varepsilon(\sigma) := \{x \in D : u_\varepsilon(x) = \sigma\}$ for $\sigma \in S^1$ are rectifiable for almost all σ and can be given a structure of integer 1-currents. Then for almost all $\sigma \in S^1$ we have

$$\partial\left[F_\varepsilon(\sigma)\right] = \partial I_\varepsilon.$$

By (3.7) we also obtain that the L^p-weak limit $\nabla(\varphi_0 - \psi)$ exists up to extracting a further subsequence, and it is equal to $\nabla^\perp u_0$, therefore again by using Sard's theorem, its divergence is the boundary of an integer 1-current I_0, which can be described using a generic fiber $F_0(\sigma)$ of u_0:

$$\mathrm{div}\nabla(\varphi_0 - \psi) = \partial I - 0.$$

Since $u_0 \in W^{1,p}$, by an easy application of the Fubini theorem to the generalized coarea formula, we have that the generic fibers $F(\sigma)$ have finite $\mathcal{H}^1$-measure, thus I_0 has finite mass.

Since we have

$$\begin{aligned} &\nabla\psi \in L^p \text{ and} \\ &\nabla^\perp u_k = \nabla(\psi - \varphi_k) \overset{L^1}{\to} \nabla^\perp u_0, \end{aligned}$$

there also holds $\nabla\varphi_k \overset{L^1}{\to} \nabla\varphi_0$, and since $\nabla\varphi_\varepsilon \overset{L^p}{\to} 0$ we have, using (3.3), that the difference of our forms is the boundary of an integer 1-current:

$$h_1 - h_2 = \partial I_0. \tag{3.8}$$

The following Lemma concludes the proof.

Lemma 3.6. *If the boundary of an integer multiplicity finite-mass 1-current I on a domain $D \subset \mathbb{R}^2$ can be represented by a L^p-function for $p \geq 1$, then $\partial I = 0$.*

Proof. Supposing that $\partial I \neq 0$ and that there exists a function such that for all $\varphi \in C^1_c(D)$ there holds

$$\langle \varphi, h \rangle = \langle \varphi, \partial I \rangle,$$

we should reach a contraddiction.

Indeed, if we take some smooth positive radial function $\varphi \in C^1_c(B_1(0))$ which is equal to 1 on $B_{1/2}(0)$ and we consider a point of approximate continuity x_0 of h such that $h(x_0) \neq 0$, then we will also have

$$\begin{aligned}\mathbb{M}(I)||\nabla\varphi||_{L^\infty} &\geq \left|\left\langle \nabla_x \left(\frac{1}{\varepsilon}\varphi(\varepsilon(x-x_0))\right), I\right\rangle\right| \\ &= \frac{1}{\varepsilon}\left|\int \varphi(\varepsilon(x-x_0))h(x)\, dx\right| \\ &\geq \frac{c\,|h(x_0)|}{\varepsilon},\end{aligned}$$

which is a contraddiction. □

4. Application of the above defined metric in the case of $D = S^2$

We hereby consider a 2-form h on $B^3 := B_1^3(0)$ such that $i^*_{\partial B^3}h = 0$ and suppose that for a fixed point $a \in B^3$ and for $0 < s' < s < \mathrm{dist}(a, \partial B^3)$ there holds

$$\forall x \in [s', s],\ \int_{\partial B_x(a)} i^*_{\partial B_x(p)}h \in \mathbb{Z}.$$

We also suppose that there exists an integral 1-current I in $[0,1]^3$ such that ∂I can be represented by $*dh$. In this case we have the following result:

Proposition 4.1. *Under the above hypotheses, for each subinterval $K \subset [s', s]$ there exists a function $M_K \in L^{1,\infty}(K, \mathbb{R})$, such that there holds*

$$[M_K(x)]^{1/p} \geq \mathrm{esssup}_{x\neq\tilde{x}\in K}\frac{d(h(x), h(\tilde{x}))}{|x-\tilde{x}|}, \tag{4.1}$$

Where the 2-form $h(x) := T_x^* i^*_{\partial B_x(a)}h$ on S^2 corresponds to the restriction $i^*_{\partial B_x(a)}h$ through the affine map $T_x : S^2 \to \partial B_x(a)$, $T_x(\theta) := a + x\theta$.

Proof. Without loss of generality, we may suppose that $s = 1$ and that a is the origin. We start by observing that given a subinterval $K' = [t, t+\delta] \subset K$, we may consider (in po0lar coordinates) a function $\bar{\varphi}(\theta, r) = \varphi(\theta)$ on $B_1(0) \setminus \{0\}$ and identify the 2-form h with the 1-form $*h$. Then for $x \in]0,1]$, $i^*_{\partial B_x(0)}h$ will be identified with a 1-form tangent to $\partial B_x(0)$, and therefore $h(x)$ is identified with n 1-form (or, after fixing the standard metric, to a 1-vector field) on S^2. Observe that

$$\begin{aligned}
\left\langle \varphi, *_{S^2} d_{S^2}\left(\int_t^{t+\delta} h(x)dx\right)\right\rangle_{S^2} &= \int_{S^2} \nabla\varphi(\theta)\cdot\left(\int_t^{t+\delta} h(x)(\theta)\, dx\right)\, d\theta\\
&= \int_t^{t+\delta}\int_{S^2} \langle d\varphi(\theta), h(x)(\theta)\rangle\, dx\, d\theta\\
&= \int_\Omega \langle d\varphi(\theta), i^*_{\partial B_x}(*h)(\theta)\rangle\, dV\\
&= \int_\Omega \langle d\bar\varphi, *h\rangle\, dV\\
&= \int_\Omega \langle \bar\varphi, *dh\rangle\, dV + \int_{\partial B_{t+\delta}} *(i^*_{\partial B_{t+\delta}} h)\bar\varphi\, d\sigma\\
&\quad - \int_{\partial B_t} *(i^*_{\partial B_t} h)\bar\varphi\, d\sigma\\
&= \int_\Omega \langle \bar\varphi, *dh\rangle\, dV + \int_{S^2} h(t+\delta)\varphi\, d\theta\\
&\quad - \int_{S^2} h(t)\varphi\, d\theta,
\end{aligned}$$

where $\Omega := B_1 \setminus B_{s'}$. We used above the definition of $h(x)$ and the fact that since $\bar\varphi$ depends only on θ we have that for any one-form ω there holds $\langle d\bar\varphi, \omega\rangle_{\partial B_x} = \langle d\bar\varphi, i^*_{\partial B_x}\omega\rangle_{\partial B_x}$. We now use the property relating the 1-current I to the form h:

$$\int_\Omega \langle\bar\varphi, *dh\rangle\, dV = \langle \bar\varphi, (\partial I)\llcorner\Omega\rangle$$

The following formula holds for C^1-approximations $\chi_\varepsilon \in C^\infty_c(]0,1[^3)$ of the characteristic function of Ω:

$$(\partial I)\llcorner\Omega = \lim_{\varepsilon\to 0}(\partial I)\llcorner\chi_\varepsilon = \lim_{\varepsilon\to 0}\left[\partial(I\llcorner\chi_\varepsilon) + I\llcorner(d\chi_\varepsilon)\right] = \partial(I\llcorner\Omega) + \lim_{\varepsilon\to 0} I\llcorner(d\chi_\varepsilon),$$

and the last term can be expressed in terms of slices along the proper function

$$\begin{aligned}
f : B_1\setminus B_{s'} &\to [s', 1]\\
(\theta, r) &\mapsto r,
\end{aligned}$$

keeping in mind that $\Omega = f^{-1}([t, t+\delta])$: we have

$$\lim_{\varepsilon\to 0} I\llcorner(d\chi_\varepsilon) = \langle I, f, t+\delta\rangle - \langle I, f, t\rangle,$$

and we observe therefore that for almost all values of $t, t+\delta$ the above contribution is an integer 0-current, and since

$$\int_{s'}^{1} \mathbb{M}\langle I, f, \tau\rangle d\tau = \mathbb{M}\left(I \llcorner f^{\#}(\chi_{[s',1]} d\tau)\right) \leq C_{s'} \mathbb{M}(I) < \infty,$$

we obtain that it has also finite mass for almost all choices of $t, t+\delta$, therefore it is a finite sum of Dirac masses with integer coefficients. we now use the following easy lemma:

Lemma 4.2. *With the above notations, if $\bar{J}$ is a finite mass rectifiable integer 1-current in $B_y \setminus B_x$ for $1 > y > x > 0$, then there exists a finite mass rectifiable integer 1-current supported on ∂B_x such that*

- *for all functions $\bar{\varphi}(\theta, r) = \varphi(\theta)\chi(r)$ such that $\chi \in C_c^\infty(]0,1])$ and $\chi \equiv 1$ on $[x, y]$ there holds $\langle \bar{\varphi}, \partial \bar{J}\rangle = \langle \varphi, \partial J\rangle$,*
- $\mathbb{M}(\bar{J}) \leq \mathbb{M}(J)$

Applying the above lemma to $\bar{J} = i \llcorner \Omega$, we obtain

$$\langle \partial(I \llcorner \Omega), \bar{\varphi}\rangle = \langle \partial J, \varphi\rangle,$$

where J is a finite mass rectifiable integer 1-current. We now can summarize what shown so far by writing (all the objects being defined on S^2)

$$\begin{aligned} *d\left(\int_t^{t+\delta} h(x)dx\right) &= h(t+\delta) - h(t) + \langle I, f, t+\delta\rangle - \langle I, f, t\rangle + \partial J \\ &= h(t+\delta) - h(t) + \sum_{i=1}^{N} d_i \delta_{a_i} + \partial J, \end{aligned}$$

Therefore we have, by definition of the metric $d(\cdot,\cdot)$, that

$$d(h(t), h(t+\delta)) \leq \left\| \int_t^{t+\delta} h(x)\, dx \right\|_{L^p(S^2)}$$

We can then further compute:

$$\begin{aligned} d(h(t), h(t+\delta)) &\leq \left[\int_{S^2} \left| \int_t^{t+\delta} h(r)(\theta)\, dr\right|^p d\theta\right]^{1/p} \\ &\leq \delta^{1-\frac{1}{p}} \left[\int_t^{t+\delta} \int_{S^2} |h(r)(\theta)|^p\, dr\, d\theta\right]^{1/p} \\ &\leq \delta \left[M_K\left(\int_{S^2} |h(\cdot)|^p\right)(t)\right]^{1/p}, \end{aligned}$$

where $M_K f$ is the uncentered maximal function of f on the interval K, defined as

$$M_K f(x) = \sup\left\{\tfrac{1}{|B_\rho(Y)|}\int_{B_\rho(Y)} |f| : \ x \in B_\rho(y) \subset K\right\}.$$

□

5. The almost everywhere pointwise convergence theorem

We next call

$$N_I h(t) := \left[M_I\left(\|h(r)\|^p_{L^p(D)}\right)(t)\right]^{1/p},$$

where $D = [0,1]^2$ or $D = S^2$.

Then the following is a restatement of the equation (4.1) in terms of $N_k h$:

$$\text{For all } x, y \in I, \text{ there holds } N_I h(x)|x-y| \geq d(h(x), h(y)). \tag{5.1}$$

We next will consider the metric space

$$Y := [L^p(D), d(\cdot,\cdot)] \cap \{h : \int_D h \in \mathbb{Z}\} \tag{5.2}$$

We then observe that $f := \left[t \mapsto \|h(t)\|^p_{L^p(D)}\right] \in L^1([s', s])$ for all $0, s', s \leq 1$, therefore we obtain by the usual Vitali covering argument for $M_I f$, that there exists a dimensional constant C for which

$$\sup_{\lambda>0} \lambda^p |\{t \in I : \ N_I h(t) > \lambda\} \leq C \int_I |f(x)| dx. \tag{5.3}$$

We can now prove the following analogous of Theorem 9.1 of [HR03] (we write down also the proof just in order to convince the reader that the hypotheses in the original statement can be changed, and in fact it is completely analogous to the original one):

Theorem 5.1. *Suppose that for each $n = 1, 2, \dots, h_n : [0,1] \to Y$ is a measurable function such that for all sbintervals $I \subset [0,1]$ there holds*

$$\sup_{\lambda>0} \lambda^p |\{t \in I \ N_I h_n(t) > \lambda\}| \leq \mu_n(I) \tag{5.4}$$

for some function $N_I h_n$ satisfying (5.1), where μ_n are positive measures on $[0,1]$ such that $\sup_n \mu_n([0,1]) < \infty$. We also suppose that we are given a functional $\mathcal{N} : Y \to \mathbb{R}^+$ which is lower semicontinuous *on Y and such that its sublevels are sequentially compact, and that for some $L \in \mathbb{R}$ there holds*

$$\sup_n \int_{[0,1]} \mathcal{N}(h_n(x))dx < L < \infty. \tag{5.5}$$

Then the sequence h_n has a subsequence that converges pointwise almost everywhere to a limiting function $h : X \to Y$ such that

$$\int_{[0,1]} \mathcal{N}(h(x))dx \leq L$$

$$\forall I \subset [0,1],\ \sup_{\lambda>0} \lambda^p |\{t \in I\ \tilde{N}_I h(t) > \lambda\} \leq \sup_n \mu_n(I),$$

where again $\tilde{N}_I h$ is a function satisfying (5.1)*.*

Remark 5.2. *In Theorem 5.1 we considered the interval* $[0,1]$ *instead of* $[s', s]$ *just for the sake of simplicity; the above results clearly extend also to the general case.*

Proof. **Claim 1.** *It is enough to find a subsequence $f_{n'}$ which is pointwise a.e. Cauchy convergent.* Indeed, in such case for a.e. $x \in [0,1]$ there will exist a unique limit $f(x) := \lim f_{n'}(x) \in \hat{Y}$, the completion of Y. For such x we can then use Fatou's lemma and (5.5), obtaining for a.e. x a further subsequence n'' (which depends on x), along which $\mathcal{N}(f_{n''}(x))$ stays bounded. By compactness of the sublevels of $\mathcal{N}$ we then have that $f(x) \in Y$.

Next, the lower semicontinuity of $\mathcal{N}$ gives us that the property (5.5) passes to the limit, while for the other claimed property we may take

$$\tilde{N}_I h(x) := \sup_{I \ni \tilde{x} \neq x} \frac{d(h(x), h(\tilde{x}))}{|x - \tilde{x}|}$$

and then use (5.1) to obtain

$$d(h(x), h(\tilde{x})) = \lim_{n'} d(h_{n'}(x), h_{n'}(\tilde{x})) \leq \liminf_{n'} N_I h_{n'}(x)|x - x'|,$$

which gives (5.4) for $\tilde{N}_I h_{n'}$, since it shows that $\tilde{N}_I h(x) \leq \liminf_{n'} N_I h_{n'}(x)$. This proves the claim.

Wanted properties. We will obtain the wanted subsequence (n') by starting with $n_0(j) = j$ and successively extracting subsequences $n_k(j)$ of $n_{k-1}(j)$. In parallel to this we will select

- countable families $\mathcal{I}_k$ of closed subintervals of $[0,1]$ which cover $[0,1]$ up to a nullset Z_k

- for $I \in \mathcal{I}_k$ we will give a point $c_I \in I$ such that $y_{j,I} := h_{n_k(j)}(c_I)$ are Cauchy sequences for all $I \in \mathcal{I}_k$ and

$$\limsup_j N_I h_{n_k(j)}(c_I) \leq \frac{1}{k|I|} \tag{5.6}$$

Claim 2. *The above choices guarantee the existence of a pointwise almost everywhere Cauchy subsequence* $h_{n'}$. Indeed, we will then take a diagonal subsequence $j' = n_j(j)$, and use the fact that the nullsets Z_k have as union a nullset Z. then clearly for $I \in \mathcal{I}_k$ with k big enough, we will have $d(f_{i'}(c_I), f_{j'}(c_I)) < \varepsilon/3$ for i', j' big enough, while for $x \in I$ we will have, by (5.6), that there exists C close to 1 such that

$$d(h_{i'}(x), h_{i'}(c_I)) \leq N_I h_{i'}(c_I)|I| \leq C\frac{1}{k}.$$

The above are enough to prove that for all $x \in [0,1] \setminus Z$ the sequence $h_{j'}$ is Cauchy, as wanted.

Obtaining the wanted properties. The subsequence $n_k(j)$ of $n_{k-1}(j)$ will be also obtained by a diagonal extraction applied to a nested family of subsequences $n_{k-1} \prec m_1 \prec m_2 \prec \ldots$ (where $a \prec b$ means that $b(j)$ is a subsequence of $a(j)$). We describe now the procedure used to pass from n_{k-1} to m_1.
We choose an integer q such that

$$q > 2k^p \sup_n \mu_n([0,1])$$

and we let $\mathcal{I}$ be the decomposition of $[0,1]$ into $2q$ nonoverlapping subintervals of equal length. Then for each n we can find q "good" intervals in $\mathcal{I}$ having μ_n-measure less than $1/(2k^p)$. The possible choices of such subsets of intervals being finite, we can find one such choice of subintervals $\{I_1, \ldots, I_q\} \subset \mathcal{I}$ and a subsequence $m_0 \succ n_{k-1}$ such that for any of these fixed good intervals and for any $j \in \mathbb{N}$, there holds

$$\mu_{m_0(j)}(I_i) < \frac{1}{2k^p}. \tag{5.7}$$

For a fixed interval I_i, we now give a name to the set of points where (5.6) is falsified at step $m_0(j)$:

$$E_{m_0(j)} := \left\{ x \in I_i : \; N_{I_i} h_{m_0(j)}(x) > \frac{1}{k|I_i|} \right\} \tag{5.8}$$

Then by (5.4), (5.7), (5.8) and since $|I_i| \leq |I| = 1$, we get

$$|E_{m_0(j)}| \leq k^p |I_i|^p \mu_{m_0(j)}(I_i) < \frac{1}{2}|I_i|^p \leq \frac{1}{2}|I_i|.$$

for j large enough, and therefore by Fatou lemma we get

$$\int_{I_i} \liminf_j \left[\chi_{E_{m_0(j)}}(x) + \frac{|I_i|}{3L}\mathcal{N}(h_{m_0(j)}(x))\right] dx \leq \frac{1}{2}|I_i| + \frac{|I_i|}{3L}L = \frac{5}{6}|I_i|$$

Therefore we can find $c_{I_i} \in I_i$ and a subsequence $m_1 \succ m_0$ so that along m_1 we have

$$\chi_{E_{m_0(j)}}(c_{I_i}) + \frac{|I_i|}{3L}\mathcal{N}(h_{m_0(j)}(c_{I_i})) < 1,$$

in particular $c_{I_i} \notin E_{m_1(j)}$, and $\mathcal{N}(f_{m_1(j)}(c_{I_i}))$ is bounded. The second fact allows us to find a Cauchy subsequence $m_2 \succ m_1$, while the first one gives us the wanted property (5.6) for I_i. We now extract such subsequences in order to obtain the same property for all the good intervals $I_1, \ldots, I_q$. These intervals cover $1/2$ of the Lebesgue measure of $[0,1]$, so we may continue the argument by an easy exhaustion, covering $[0,1]$ by "good" intervals up to a set of measure zero. □

6. Verification of the properties needed in Theorem 5.1

We have seen that the functions $N_I h_n$ defined in Section 5 satisfy the hypotheses (5.1) and (5.4), where by (5.3) it is clear that we can choose

$$\mu_n(I) := C\int_I \|h_n(x)\|^p_{L^p(D^2)}dx.$$

In order to use the abstract theorem 5.1, we will now show that the functional $\mathcal{N} : Y \to \mathbb{R}^+$ defined by

$$\mathcal{N}(h) := \int_D |h|^p dx \tag{6.1}$$

where D is a 2-dimensional domain (for example $[0,1]^2$ or S^2) and

$$Y := \{h \in L^p(D, \wedge^2 D) : \int_D h \in \mathbb{Z}\}, \tag{6.2}$$

is a metric space with the distance d (this was proved in Proposition 3.3) satisfies the properties stated in Theorem 5.1, namely that it is sequentially lower semicontinuous and that it has sequentially compact sublevels. The proofs are given in the following two propositions.

Proposition 6.1. *Under the notations* (6.1) *and* (6.2)*, the functional* $\mathcal{N} : Y \to \mathbb{R}^+$ *is sequentially lower semicontinuous.*

Proof. In other words, we must prove that whenever $h_n \in Y$ is a sequence such that for some $h_\infty \in Y$ there holds

$$d(h_n, h_\infty) \to 0, \tag{6.3}$$

we also have

$$\liminf_{n\to\infty} \mathcal{N}(h_n) \geq \mathcal{N}(h_\infty). \tag{6.4}$$

We may suppose that $\mathcal{N}(h_n)$ are bounded, therefore h_n are bounded in L^p so up to extracting a subsequence we have

$$h_n \overset{L^p}{\rightharpoonup} k_\infty,$$

for some $k_\infty \in L^p$. By taking the test function $f \equiv 1$, which is in the dual space L^q since D is bounded, we also obtain that $k_\infty \in Y$. Up to a subsequence we may thus assume that for all n we have $\int_D h_n = \int_D h_\infty \in \mathbb{Z}$. By the lower semicontinuity of the norm with respect to weak convergence, we have:

$$\liminf_{n\to\infty} \mathcal{N}(h_n) \geq \mathcal{N}(k_\infty),$$

therefore it suffices to prove

$$h_\infty = k_\infty, \tag{6.5}$$

in order to have (6.4), and thus to conclude. We now write (6.3) using the definition of d: there must exist finite mass integer 1-currents I_k and vectorfields X_k converging to zero in L^p such that

$$h_k - h_\infty = \mathrm{div} X_k + \partial I_k + \delta_0 \int_D (h_k - h_\infty) = \mathrm{div} X_k + \partial I_k.$$

Now we proceed as before, and we define ψ_k and φ_k by

$$\begin{cases} h_k - h_\infty = \Delta\psi_k, \int_D \psi_k = 0 \\ \Delta\varphi_k = \mathrm{div} X_k, \end{cases}$$

therefore having $\mathrm{div}(\nabla(\psi_k - \varphi_k)) = \partial I_k$. We also have that $\nabla\varphi_k \to 0$ in L^p and $\nabla\psi_k$ is bounded in $W^{1,p}$, thus up to extracting a subsequence we may assume that

$$\nabla\psi_k \overset{W^{1,p}}{\rightharpoonup} \nabla\psi_\infty.$$

Now we can use Proposition 3.5 in order to write

$$\nabla(\psi_k - \varphi_k) = \nabla^\perp u_k$$

for functions $u_k \in W^{1,p}(D, \mathbb{R}/2\pi\mathbb{Z})$ such that $\|\nabla u_k\|_{L^p} \leq C$. Up to extracting a subsequence we therefore have $\nabla u_k \rightharpoonup \nabla u_\infty$ weakly in L^p, thus also in L^1_{loc}, and in particular we have

$$\nabla\psi_\infty = \nabla^\perp u_\infty.$$

Since weak-$W^{1,p}$-convergence implies $\mathcal{D}'$-convergence, we have as in the proof of Proposition 3.3 that

$$\partial I_k \overset{\mathcal{D}'}{\to} \partial I_\infty + \mathrm{div}(\nabla^\perp u_\infty) = \mathrm{div}\nabla\psi_\infty,$$

where I_∞ is an integer finite mass 1-current. By Lemma 3.6 we have than that $\partial I_\infty = 0$, which implies that

$$h_k - h_\infty \stackrel{\mathcal{D}'}{\to} 0,$$

Therefore we have (6.5), which concludes the proof. □

Proposition 6.2. *Under the notations* (6.1) *and* (6.2)*, and for any* $C > 0$*, the set* $\{h \in Y : \mathcal{N}(h) \leq C\}$ *is* d*-sequentially compact.*

Proof. We must prove that whenever we have a sequence h_n such that $\|h_n\|_{L^p}$ is bounded, then up to extracting a subsequence we have that for some $k_\infty \in Y$ there holds

$$d(h_n, k_\infty) \to 0 \tag{6.6}$$

We again obtain, that we can take a subsequence of the h_n which is weakly-L^p-convergent to a function $k_\infty \in L^p$. Then, as in the proof of Proposition 6.1 we have $\int_D k_\infty \in \mathbb{Z}$ and up to extracting a subsequence we may assume that $\int_D (h_n - k_\infty) = 0$ for all n. Then we define ψ_n to be the solution of

$$\begin{cases} \Delta\psi_n = h_n - k_\infty \\ \int_D \psi_n = 0, \end{cases}$$

and we claim that there holds

$$\|\nabla\psi_n\|_{L^p} \to 0. \tag{6.7}$$

This is then enough to conclude, since we can then set $X_n = \nabla\psi_n$ which gives an upper bound of $d(h_n, k_\infty)$ which converges to zero, proving (6.6).

In order to prove (6.7) we start by expressing

$$\nabla\psi_n(x) = \int_D \nabla G(x, y)\,[h_n(y) - k_\infty(y)]\,dy,$$

where G is the Green function of D. We know that $\nabla G \in L^q$ for all $q < 2$ and we also have that $h_n - k_\infty$ is a function converging to zero weakly in L^p and which is bounded in L^p. From the weak convergence we then obtain the pointwise convergence

$$\nabla\psi_n(x) \to 0 \text{ for all } x. \tag{6.8}$$

We can then use the L^p-boundedness of $h_n - k_\infty$ together with the Young inequality

$$\|\nabla\psi_n\|_{L^r} \leq \|\nabla G\|_{L^q}\|h_n - k_\infty\|_{L^p},$$

for q as above. We then have that $\|\nabla\psi_n\|_{L^r}$ are bounded once the following equivalent relations hold:

$$\frac{1}{r} > \frac{1}{p} + \frac{1}{2} - 1 \Leftrightarrow r < \frac{2p}{2-p},$$

In particular we have the boundedness in L^r for some $r > p$. This together with the pointwise convergence (6.8) and with the L^p-boundedness gives then (6.7), as wanted. □

7. Proof of Theorem 2.7

Lemma 7.1. *For all $\alpha, \varepsilon > 0$ there exists a subset $E_{\alpha,\varepsilon} \subset I$ such that $|E_{\alpha,\varepsilon}| < \varepsilon$ and that there exists $N_{\alpha,\varepsilon}$ such that $n > N_{\alpha,\varepsilon}$ and $t \in E_{\alpha,\varepsilon}$ imply*

$$d(h_n(t), h_\infty(t)) < \alpha$$

Proof. Call $E_{m,n} := \{x \in I : \, d(h_i(x), h_\infty(x)) \leq 1/m \text{ for } i \geq n\}$. Then for fixed $m_\alpha > \alpha^{-1}$, the sets $E_{m_\alpha,n}$ form an increasing sequence whose union is I. It follows that $|E_{m_\alpha,n}| \to |I|$, so we find $N_{\alpha,\varepsilon}$ such that $|I \setminus E_{m_\alpha,N_{\alpha,\varepsilon}}| \leq \varepsilon$. We then choose $E_{\alpha,\varepsilon} := E_{m_\alpha,N_{\alpha,\varepsilon}}$. It is easy to verify that this set is as wanted. □

Lemma 7.2. *Fix $x \in I$ and a 2-form $h_\infty(x)$. For all $c > 0$ there exists $\varepsilon > 0$ such that*

$$\left.\begin{array}{r} d(h(x), h_\infty(x)) < \alpha \\ \int |h(x)|^p \leq A \end{array}\right\} \Rightarrow \int h(x) = \int h_\infty(x).$$

Proof. Suppose by contradiction that there exists a $A > 0$ such that for all $k \in \mathbb{N}$ there exists h_k such that

$$\begin{aligned} d(h_k(x), h_\infty(x)) &\leq \frac{1}{k} \\ \int |h_k(x)|^p &\leq A \\ \int h_k(x) &\neq \int h_\infty(x) \end{aligned}$$

By the second property, we can extract a subsequence $h_{k'}(x)$ of the $h_k(x)$ converging weakly in L^p. In particular we would then have

$$\mathbb{Z} \ni \int h_{k'}(x) \to \int h'_\infty(x)$$

In particular, for some $N \in \mathbb{N}$ large enough, the subsequence $h_{k''} := h_{k'+N}(x)$ satisfies

$$\int h_{k''}(x) = \int h'_\infty(x).$$

We now prove that $h_\infty(x) = h'_\infty(x)$. For this it is enough to prove that $h_{k''}(x) \stackrel{d}{\to} h'_\infty(x)$. and this follows exactly as in the proof of Proposition 6.2. We thus contradicted the assumption $\int h_k(x) \neq \int h_\infty(x)$, as wanted. □

Proof of Theorem 2.7. By the L^p-boundedness of the F_n, it is clear that we may find a weakly converging subsequence $F_{n'} \stackrel{L^p}{\rightharpoonup} F_\infty$. We suppose by contradiction that there exists a point $x \in B_1(0)$ and two radii $0 < s' < s < \mathrm{dist}(x, \partial B_1(0))$ such that

$$\exists S \subset [s', s] \text{ s.t. } \mathcal{H}^1(s) > 0 \text{ and } \forall t \in S, \quad \int_{\partial B_t(x)} i^*_{\partial B_t(x)} F \notin \mathbb{Z}. \tag{7.1}$$

We then identify the forms given by

$$\tilde{F}_n|_{\partial B_r(x)} := i^*_{\partial B_r(x)} F \text{ for } r \in [s', s],$$

with functions (defined almost everywhere) $h_n : [s', s] \to Y$ (with the notations of Section 5). By Theorem 5.1, we can assume (up to extracting a subsequence) that there exists h_∞ such that for almost all $t \in I$ there holds $d(h_n(t), h_\infty(t)) \to 0$.

We call

$$\left| \int h_n(x) - \int h_\infty(x) \right| := f_n(t).$$

Since we have $f_n \geq 0$, if we prove that the f_n converge in L^1-norm, then the almost everywhere pointwise convergence follows, implying the fact that $|S| = 0$ and reaching the wanted contradiction. To prove Theorem 2.7 we therefore have to prove that

$$\lim_{n\to\infty} \int f_n(t) dt = 0. \tag{7.2}$$

We start by calling

$$F_{n,A} := \left\{ t \in I : \int |h_n(t)|^p \geq A \right\}.$$

It clearly follows that (with C as in the statement of the theorem)

$$|F_{n,A}| \leq \frac{1}{A} \int \left(\int |h_n(t)|^p \right) dt = \frac{C}{A}$$

Now take A such that the above quantity is smaller than ε, and use Lemma 7.2 to obtain a constant α such that $d(h_n(t), h_\infty(t)) < \alpha$ implies $f_n(t) = 0$ for t such that $\int |h_n(t)|^p < A$, i.e. for $t \notin F_{n,A}$. Now with the above choice of α apply Lemma 7.1 and obtain a set $E_{\alpha,\varepsilon}$ so that $|I \setminus E_{\alpha,\varepsilon}| < \varepsilon$ and an index $N_{\alpha,\varepsilon}$ such that for $n \geq N_{\alpha,\varepsilon}$ and for

$t \in E_{\alpha,\varepsilon}$ there holds $d(h_n(t),0) < \alpha$, and therefore $f_n(t) = 0$.

For $n > N_{\alpha,\varepsilon}$, the function $f_n(t)$ can therefore be nonzero only on $E_{\alpha,\varepsilon} \cup F_{n,A}$, and we have

$$\begin{aligned}\int f_n(t)dt &\leq \int_{E_{\alpha,\varepsilon}\cup F_{n,A}} f_n(t)dt \\ &\leq |E_{\alpha,\varepsilon} \cup F_{n,A}|^{1-1/p}\left[\int |f_n(t)|^p dt\right]^{1/p} \\ &\leq (2\varepsilon)^{1-1/p}C,\end{aligned}$$

whence the claim (7.2) follows, finishing the proof of our result. □

8. The case $p = 1$

We prove here the result stated in the Main Threorem 1.1 for $p = 1$. We consider the case when the domain is $[0,1]^3$ for simplicity. The case of general domains is totally analogous.

Proposition 8.1. *Consider a signed Radon measure $X \in \mathcal{M}^3([0,1]^3)$, with total variation equal to 1. Then there exists a family of vectorfields $X_k \in L^1_{\mathbb{Z}}$ such that*

(1) *There are two constants $0 < c < C < \infty$ such that*
$$\begin{cases} \forall k \quad c < \|X_k\|_{L^1([0,1]^3)} < C \\ \mathbb{M}(\mathrm{div}X_k) \to \infty \end{cases}$$
(2) $\mathrm{div}X_k = \partial I_k$ *for a sequence of integer rectifiable currents I_k of bounded mass, and finally*
$$X_k \rightharpoonup X$$

From the above, it immediately follows:

Corollary 8.2. *The class $\mathcal{F}^1_{\mathbb{Z}}$ is not closed by weak convergence.*

The following holds for all $p < \frac{n}{n-1}$ in n dimensions:

Lemma 8.3. *Given a segment $[a,b] \subset \mathbb{R}^n$ of length $\varepsilon > 0$ and a number $\delta > 0$, if $p < \frac{n}{n-1}$ then it is possible to find a vectorfield $X \in L^p(\mathbb{R}^n,\mathbb{R}^n)$ with*

$$\begin{aligned}\mathrm{div}X &= \delta_a - \delta_b, \\ \mathrm{spt}X &\subset [a,b] + B_\varepsilon(0), \\ \|X\|_{L^p} &\leq C\varepsilon^{n-(n-1)p}\end{aligned}$$

where C is a geometric constant, and for two sets A, B, we denote $A + B := \{a + b : a \in A,\ b \in B\}$.

Proof. We may suppose that $a = (-\varepsilon, 0, \ldots, 0), b = (\varepsilon, 0, \ldots, 0)$ first. We then define the piecewise smooth

$$\begin{cases} X(\pm\varepsilon(t-1), \varepsilon st) &= \left(\frac{1}{\varepsilon t^{n-1}|B_1^{n-1}|}, \pm\frac{s}{\varepsilon t^{n-1}|B_1^{n-1}|}\right) \text{ for } (t,s) \in [0,1] \times B_1^{n-1} \\ X(x,y) &= (0,0) \text{ if } |x| + |y|_{\mathbb{R}^{n-1}} > \varepsilon. \end{cases}$$

Then clearly $\mathrm{spt}X \subset [a,b] + B_\varepsilon$, and using the divergence theorem it is also easily shown that $\mathrm{div}X = \delta_a - \delta_b$ in the sense of distributions. For the last estimate, we observe that

$$|X(x,y)| \leq \frac{C}{(\varepsilon - |x|)^{n-1}} \chi_{\{|x|+|y|\leq\varepsilon\}}(x,y),$$

so we can estimate

$$\begin{aligned} \int_{\mathbb{R}^2} |X|^p \, dx \, dy &\leq C \int_0^\varepsilon \frac{(\varepsilon - x)^{n-1}}{(\varepsilon - x)^{(n-1)p}} \, dx \\ &= C\varepsilon^{n-(n-1)p} \end{aligned}$$

□

Proof of Proposition 8.1. We will do our construction first in the simpler model case $F = dy \wedge dz \llcorner [0,1]^3$. The modifications leading to the general case are treated separately.

- *The case of* $X \equiv (1,0,0)$. We consider the collections of segments in $[0,1]^3$ given by

$$\mathcal{S}_k := \left\{ \left[\left(-2^{-3k-1}, 0, 0\right), \left(2^{-3k-1}, 0, 0\right) \right] + (a,b,c) \ : \ (a,b,c) \in 2^{-k}\mathbb{Z}^3 \cap]0,1[^3 \right\}.$$

  We then define an integral rectifiable 1-current I_k as the canonical integration from right to left along all the segments of $\mathcal{S}_k$. There clearly holds

$$\mathbb{M}(I_k) = 2^{-3k}(2^k - 1)^3 \to 1, \tag{8.1}$$

  and it is a standard exercise in geometric measure theory (based on the approximation of $\mathcal{H}^3 \llcorner [0,1]^3$ by sums of Dirac measures in the points $2^{-k}\mathbb{Z}^3 \cap]0,1[^3$) to show that there holds:

$$I_k \rightharpoonup \mathcal{H}^3 \llcorner [0,1]^3 \otimes dx \simeq (1,0,0). \tag{8.2}$$

  We can then use Lemma 8.3 for each one of the segments in $\mathcal{S}_k$ and with $\delta = \frac{1}{2}\varepsilon = 2^{-3k}$ (which produces a set of $(2^k - 1)^3$ vectorfields with disjoint supports, which can then be consistently extended to zero outside the set of the supports) each of whose L^1-norms is equal to $C\varepsilon^2 \max\{\delta^{-1}, \varepsilon^{-1}\} = 2^{-3k}C$, which is proportional to the mass of the respective segment. Therefore (using (8.2)), property (1) follows.

The last point of the proposition follows by proving that also the vectorfields X_k converge as 1-currents to the diffuse current X. The strategy used is as the one usually adopted for the proof of the converence of the I_k: for a fixed smooth vectorfield a and for $k \to \infty$ we may approximate

$$\begin{aligned}
\langle X_k, a\rangle &:= \sum_{\sigma\in\mathcal{S}_k}\int_{\mathrm{spt}X^\sigma} X_k\cdot a\\
&= \sum_{P\in\mathbb{Z}^3\cap]0,1[^3}\left[\left(\int_{\mathrm{spt}X^\sigma} X_k(x)dx\right)\cdot a(P)+\right.\\
&\quad\left.+\int_{\mathrm{spt}X^\sigma} X_k(x)\cdot Da(P)[x-P]dx\right]+O_a(2^{-3k})\\
&= \sum_{P\in 2^{-k}\mathbb{Z}^3\cap]0,1[^3} 2^{-3k}(1,0,0)\cdot a(P)+O_a(2^{-3k})\\
&\to \int_{[0,1]^3} a(x)\cdot(1,0,0)dx
\end{aligned}$$

where the integral containing the differential Da is zero by the symmetry properties of X^σ and using the fact that

$$\begin{aligned}
\frac{|O_a(\varepsilon)|}{\varepsilon} &\leq \sup\left\{\left|\frac{a(x+\varepsilon u)-a(x)}{\varepsilon}-Da(x)[u]\right| : x\in B^3_1,\ u\in S^2\right\}\\
&\to 0 \text{ as } \varepsilon\to 0.
\end{aligned}$$

- *The case of $X=(\rho,0,0)\in\mathcal{M}^3([0,1]^3)$, where ρ is a probability density on* $[0,1]^3$. In this case we consider the 2^{3k} disjoint cubes $\mathcal{C}_k$ having the same centers as the segments in $\mathcal{S}_k$ and sidelength 2^{-k}, and in the above construction we substitute to the segment $\sigma_k\in\mathcal{S}_k$ the segment σ'_k having the same center, but length equal to $\rho(C_k)$, where $C_k\in\mathcal{C}_k$ is the cube with center equal to the one of σ_k and σ'_k. The newly obtained currents I'_k will still satisfy (8.1) and the analogous of (8.2) given by:
$$I'_k \rightharpoonup \rho\otimes dx \simeq (\rho,0,0).$$
It is then easy to apply suitable modifications to the above proof showing that also in this case property (2) holds.
- *The general case.* We can write (by Radon-Nikodym decomposition):
$$X=(\rho_1^+-\rho_1^-,\rho_2^+-\rho_2^-,\rho_3^+-\rho_3^-),$$
where ρ_i are positive Radon measures of mass less than 1. Doing separately the construction in the previous point for all

the ρ_i we obtain integer rectifiable currents I_k of mass bounded by 6, each of which is supported on finitely many segments. Applying Lemma 8.3 to each of the above segments, we obtain vectorfields converging as before to the measure X, and since the supports of the vectorfields obtained in this way superpose not more than 6 times, the estimate of the Lemma (used here for $p=1$) still holds, up to changing the constant.

□

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