A sharp non-linear Gagliardo–Nirenberg type estimate and applications to the regularity of elliptic systems

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Abstract

We prove the inequality

$$\int |\nabla u|^{s+2}\, dx \leq C(n,s)\|u\|^2_{BMO} \int |\nabla u|^{s-2}|\nabla^2 u|^2\, dx, \qquad s \geq 2,$$

and, to give a sample of possible applications, show how it can be used to obtain ε-regularity results for solutions of a wide class of nonlinear degenerate elliptic systems

$$-\mathrm{div}\left(|\nabla u|^{p-2}\nabla u\right) = G(x,u,\nabla u),$$

where G grows like $|\nabla u|^p$.

1 Introduction

The seminal paper [3] of Coifman, Lions, Meyer and Semmes has triggered numerous new applications of Hardy spaces and the space BMO of functions of bounded mean oscillation to nonlinear partial differential equations. These applications include, among other things, various problems of variational and geometric origin: harmonic and p-harmonic maps, H-systems, wave maps etc., see e.g. [1], [6], [17], [18], [30] and *many* others. In a vague sense, a common feature of all those works is that the duality of BMO and the Hardy space $\mathcal{H}^1$ is much more subtle than the duality of L^∞ and L^1. In many instances the local BMO norm of a function or mapping is small while the L^∞ norm is just finite. Moreover, in the critical case $kp = n$ of Sobolev imbedding theorem $W^{k,p}$ is not contained in L^∞, but we have instead $W^{k,p} \subset BMO$ (in fact, $W^{k,p} \subset VMO$, i.e. Sobolev functions have vanishing mean oscillation in the borderline case; this follows easily from Poincaré inequality).

Recently, Y. Meyer and the first author of this paper have proved the interpolation inequality

$$\|\nabla f\|^2_{L^4(\Omega)} \leq C(\Omega)\|f\|_{BMO(\Omega)}\|f\|_{W^{2,2}(\Omega)}, \tag{1.1}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded smooth domain (see also [2] for related improved version of Sobolev inequality). This inequality is sharper than the corresponding case of the classical Gagliardo–Nirenberg inequality

$$\|\nabla f\|^2_{L^4}(\Omega) \leq C(\Omega)\|f\|_{L^\infty(\Omega)}\|f\|_{W^{2,2}(\Omega)};$$

this is why it was possible to use (1.1) to obtain the main result of [20], a theorem on partial regularity of a class of Yang–Mills connections.

The proof of (1.1) in [20] employs Littlewood–Paley decomposition, interpolation in Triebel–Lizorkin spaces F^s_{pq}, and the link between BMO and homogeneous Besov spaces. D. Pumberger [23] generalized this argument to show that

$$\|\nabla f\|^2_{L^{2p}(\Omega)} \le C(p,\Omega)\|f\|_{BMO(\Omega)}\|f\|_{W^{2,p}(\Omega)}, \qquad p>1. \tag{1.2}$$

He also used (1.2) to prove that the singular set $\operatorname{Sing} u$ of a stationary harmonic map $u \in W^{1,p}(\Omega,\mathcal{N})$, $p>2$, satisfies $H^{n-p}(\operatorname{Sing} u)=0$.

Different, simpler proofs of (1.1) and (1.2) have been given in [29], where the argument is based on the Fefferman duality theorem $(\mathcal{H}^1)^* = BMO$ and the well known trick used by Coifman et al. [3] to prove that $\det Df \in \mathcal{H}^1$ for every mapping $f \in W^{1,n}(\mathbb{R}^n,\mathbb{R}^n)$.

In this paper, we obtain another interpolation inequality of this type. In the simplest case of compactly supported smooth functions on $\mathbb{R}^n$ it reads

$$\int_{\mathbb{R}^n} |\nabla u|^{s+2}\,dx \le C(n,s)\|u\|^2_{BMO}\int_{\mathbb{R}^n} |\nabla u|^{s-2}|\nabla^2 u|^2\,dx, \qquad s\ge 2. \tag{1.3}$$

A more precise local variant, which forms one of the main results of this paper, is stated below, in Section 3.

We also give one application of this inequality to nonlinear elliptic systems of the form

$$-\operatorname{div}\left(|\nabla u|^{p-2}\nabla u^i\right) = G^i(x,u,\nabla u), \qquad i=1,\dots,N, \tag{1.4}$$

where $u=(u^1,\dots,u^N)\colon \mathbb{R}^n \supset \Omega \to \mathbb{R}^N$, and $p>2$. Throughout the whole paper we assume that

$$G\colon \Omega\times\mathbb{R}^N\times\mathbb{R}^{nN}\to\mathbb{R}^N$$

is of class C^1 and satisfies, for some constant Λ, the growth condition

$$|G(x,u,\nabla u)| \le \Lambda|\nabla u|^p \tag{1.5}$$

(i.e., G grows critically with the gradient of u).

This is a wide class of systems that includes p-harmonic maps into compact Riemannian manifolds and H-systems in higher dimensions.

We are interested in the so-called ε-regularity of those weak solutions u that belong to $W^{2,p}$ *and* have bounded mean oscillations. (These assumptions are quite strong. However, note that we just assume (1.5); no other structure conditions are imposed.)

It is well known that for numerous nonlinear elliptic problems "small scaled energy of a solution u [on some ball] implies its regularity [on a smaller ball]", see e.g. Evans [6] and Bethuel [1] for stationary harmonic maps, or Toro and Wang [30], and [28] for stationary p-harmonic maps. In these papers, smallness of energy yields, via the monotonicity formulae, smallness of BMO norm of u, and this in turn leads to Morrey type inequalities for the gradient. One also has, however, to show that the right hand side of the system has special, determinant-like structure, and this is usually a nontrivial task. For $p = 2$ this amounts to the construction of a Coulomb moving frame; for $p \neq 2$, due to this difficulty, partial regularity results for stationary p-harmonic maps are available only for symmetric targets (see Section 1.1 below).

Our aim is to show that ε-regularity results can be obtained for the general system (1.4) in a relatively simple way *once* it is known that the solution $u \in W^{2,p} \cap BMO$. No extra information on the structure of the nonlinear term $G(x, u, \nabla u)$ is necessary. Inequality (1.3) allows one to control the influence of the critical term.

Here are the precise statements of these ε-regularity results.

Theorem 1.1 *Assume that $u \in W^{2,p}(\Omega, \mathbb{R}^N) \cap BMO(\Omega, \mathbb{R}^N)$ is a weak solution of* (1.4).

For each $q \in (1, \infty)$ there exists a constant $\varepsilon_0 = \varepsilon_0(n, N, p, q, \Lambda)$ such that if a ball $B_\varrho \subset \Omega$ and $\|u\|_{BMO(B_\varrho)} < \varepsilon_0$, then

$$\left(\left(\frac{\varrho}{2}\right)^q \fint_{B_{\varrho/2}} |\nabla u|^q \, dx\right)^{1/q} \leq C(n, N, p, \Lambda)\left(\varrho^p \fint_{B_\varrho} |\nabla u|^p \, dx\right)^{1/p}. \tag{1.6}$$

Theorem 1.2 *Assume that $u \in W^{2,p}(\Omega, \mathbb{R}^N) \cap BMO(\Omega, \mathbb{R}^N)$ is a weak solution of* (1.4).

There exists a constant $\varepsilon_1 = \varepsilon_1(n, N, p, \Lambda)$ such that if $B_\varrho \subset \Omega$ and

$$\max\left\{\|u\|_{BMO(B_\varrho)}, \left(\varrho^p \fint_{B_\varrho} |\nabla u|^p \, dx\right)^{1/p}\right\} < \varepsilon_1,$$

then $|\nabla u| \in L^\infty(B_{\varrho/4})$ and

$$\operatorname*{ess\,max}_{B_{\varrho/4}} |\nabla u| \leq \frac{C(n, N, p, \Lambda)}{\varrho}\left(\varrho^p \fint_{B_\varrho} |\nabla u|^p \, dx\right)^{1/p}. \tag{1.7}$$

These two theorems are closely related to recent results of Duzaar and Mingione [5] who generalize to the p-harmonic case Simon's approach to the regularity theory of harmonic maps, and prove the p-harmonic approximation lemma. There are some notable differences between their and our approach. In [5, Sections 3 and 4] the authors prove Hölder continuity of a solution u assuming that $u \in W^{1,p}$ has small excess $E(u, B_\varrho) = \varrho^{-n} \int_{B_\varrho} |u -_{B_\varrho}|^p \, dx$, and that a Caccioppoli (= reverse Poincaré) inequality is satisfied. The latter condition can be derived for *minimizing* p-harmonic maps. We require more, namely the existence of $D^2u \in L^p$, and the smallness condition is also a bit stronger. However, due to the strength of inequality (1.3), Theorems 1 and 2 can be applied to *arbitrary* — not just minimizing — p-harmonic maps in $W^{2,p}$.

Let us explain this in some detail below.

1.1 Partial regularity of p-harmonic maps in $W^{2,p}$

Let $\mathcal{N}$ be a compact d-dimensional Riemannian manifold isometrically embedded in $\mathbb{R}^N$. Recall that $u \in W^{1,p}(\Omega, \mathcal{N})$ is *weakly p-harmonic* iff u is a critical point of $E_p(u) := \int_\Omega |\nabla u|^p \, dx$ with respect to variations in the range, i.e.

$$\left.\frac{d}{dt}\right|_{t=0} E_p\Big[\pi \circ (u + t\varphi)\Big] = 0 \qquad \text{for each } \varphi \in W_0^{1,p}(\Omega, \mathbb{R}^N) \cap L^\infty(\Omega, \mathbb{R}^N), \tag{1.8}$$

where π stands for the nearest point projection of a tubular neighbourhood of $\mathcal{N}$ onto $\mathcal{N}$. A standard computation (see e.g. Fuchs [12] or Hélein [18]) shows that (1.8) reads

$$-\mathrm{div}\,(|\nabla u|^{p-2}\nabla u) \perp T_u\mathcal{N} \qquad \text{in the sense of } \mathcal{D}'(\Omega, \mathbb{R}^N), \tag{1.9}$$

or equivalently

$$-\mathrm{div}\,(|\nabla u|^{p-2}\nabla u) = |\nabla u|^{p-2}A(u)(\nabla u, \nabla u), \tag{1.10}$$

where A denotes the second fundamental form of $\mathcal{N} \subset \mathbb{R}^N$. A p-harmonic map u is said to be *stationary* iff it is a critical point of E_p w.r.t. variations of the domain.

Maps that minimize E_p are regular (i.e., C^∞ when $p = 2$ and $C^{1,\alpha}$ when $p \neq 2$) outside a closed singular set $\mathrm{Sing}\, u$ which has Hausdorff dimension at most $m - [p] - 1$; see Schoen and Uhlenbeck [25] for $p = 2$, and Hardt and Lin [13], Fuchs [9], [10] , and Luckhaus [19] for $p \neq 2$ (in this case Duzaar and Mingione [5] offer a new, clear proof). The estimate of the dimension of $\mathrm{Sing}\, u$ is sharp.

The assumption that u be a local minimum of energy is crucial: there are examples of harmonic maps $u : \mathbb{B}^3 \to S^2$ that are nowhere continuous, see [24].

For $p = 2$, *all* weakly harmonic maps into arbitrary compact Riemannian manifolds are smooth. This was been proved by Hélein in a series of papers [14], [15], [16] (see also his book [18] and the survey [17]), first for $\mathcal{N} = \mathbb{S}^{n-1}$, then for $\mathcal{N}$ being a homogeneous space, and finally for arbitrary compact targets. The proof was based on the duality of Hardy space and BMO. Extending Hélein's methods, Evans [6] (for round spheres) and then Bethuel [1] (for arbitrary targets) have proved that the $(n-2)$-dimensional Hausdorff measure of the singular set of a stationary harmonic map is equal to zero.

These 'harmonic' results have been extended by various authors to the case of $p \neq 2$. If $u \in W^{1,p}(\Omega, \mathcal{N})$ is a stationary weakly p-harmonic map, then $H^{n-p}(\mathrm{Sing}\, u) = 0$. This result has been proved — but only for symmetric targets $\mathcal{N}$, i.e. round spheres or, more generally, homogeneous spaces — by various authors, see e.g. [11], [22], [27], [28], [30]; Toro and Wang [30] give the most general version. It remains an open question whether the result holds for general targets.

However, we have the following theorem.

Theorem 1.3 *Let $\Omega \subset \mathbb{R}^n$ and let $\mathcal{N}$ be a d-dimensional compact Riemannian manifold isometrically embedded in $\mathbb{R}^N$. Suppose that $2 < p \leq \frac{n}{2}$ and $u \in W^{2,p}(\Omega, \mathcal{N})$ is a p-harmonic map. Then u is locally Lipschitz on an open subset $V \subset \Omega$ such that $\mathcal{H}^{n-2p}(\Omega \setminus V) = 0$. (If $p > n/2$, then $V \equiv \Omega$.)*

Crucial difficulties of the proof are covered by Theorems 1.1 and 1.2; other ingredients are well known to experts. Thus, we only give a brief

Sketch of the proof of Theorem 1.3. Recall that by standard Gagliardo–Nirenberg inequalities $\nabla u \in L^{2p}$ for each $u \in W^{2,p}(\Omega, \mathcal{N})$. Thus, by Hölder inequality, the set

$$S_1 = \Big\{ x \in \Omega \colon \lim_{r \to 0} r^p \fint_{B(x,r)} |\nabla u|^p \, dy = 0 \Big\}$$

contains

$$S_2 = \left\{ x \in \Omega\colon\ \lim_{r\to 0} r^{2p} \fint_{B(x,r)} |\nabla u|^{2p}\, dy = 0 \right\},$$

and $H^{n-2p}(\Omega \setminus S_2) = 0$ by Frostman's lemma, see Ziemer [32, Cor. 3.2.3]. On some ball around each point in S_1 the map u is locally Lipschitz by Theorem 1.2. Indeed, on some ball around each point in S_1 the scaled energy is small. Since second derivatives of u exist in L^p, one can integrate (1.10) against test vectors $\zeta \frac{\partial u}{\partial x_j}$ to obtain

$$\int_\Omega |\nabla u|^p \mathrm{div}\, \Phi\, dx = p \sum_{j,k,l} \int_\Omega |\nabla u|^{p-2} \frac{\partial u^j}{\partial x_k} \frac{\partial u^j}{\partial x_l} \frac{\partial \Phi_l}{\partial x_k}\, dx \qquad \text{for all } \Phi \in C_0^\infty(\Omega, \mathbb{R}^n).$$

Thus, the monotonicity formula is automatically satisfied, $u \in BMO$ and its local BMO norm is controlled by the scaled energy; see e.g. Evans [6] (for $p \neq 2$ the computation is identical).

Moreover, one shows in a standard way S_1 is open, and it is enough to take $V = S_1$. Finally, $H^{n-2p}(\Omega \setminus S_1) \leq H^{n-2p}(\Omega \setminus S_2) = 0$.

The rest of the paper is organized as follows. In Section 3, we we derive a rather standard Caccioppoli inequality for the gradient of weak solutions of (1.4). Section 3 contains the proof of the interpolation inequality (1.3) and its local variant which is tailored suitably for our purposes. Finally, in Section 4 we show how to combine (1.3) and the Caccioppoli inequality to prove Theorems 1.1 and 1.2 via a modified version of Moser's iterative method [21]. (This approach to gradient bounds for systems involving the p-Laplacian was used by Uhlenbeck [31], and also by DiBenedetto and Friedman [4] in the parabolic context; in our setting the critical nonlinearity G leads to new difficulties which were absent in those two papers.)

Notation. Barred integrals denote averages, i.e. $\fint_A f\, dx = |A|^{-1} \int_A f\, dx$; sometimes we also write $f_A = \fint_A f\, dx$. $BMO(\mathbb{R}^n)$ stands for the space of functions of bounded mean oscillation, see e.g. [26, Chap. 4], with the seminorm

$$\|f\|_{BMO}\colon = \sup_Q \left(\fint_Q |f(y) - f_Q|\, dy \right) < +\infty,$$

the supremum being taken over all cubes Q in $\mathbb{R}^n$. (One can replace the average f_Q by any other constant c_Q; this does not affect the definition.) Primes are used to denote Hölder conjugate exponents, i.e. $p' = p/(p-1)$ for $p \geq 1$ etc. Finally, the letter C stands for a general constant that may change its value even in a single string of estimates. Numbered constants C_i depend only on n, N, p and the growth constant Λ.

2 Caccioppoli inequality

In this Section we prove the following Caccioppoli inequality for derivatives of solutions of (1.4).

Lemma 2.1 *Assume that $u \in W^{2,p}$ solves (1.4). Let $\zeta \in C_0^\infty(\Omega)$ and set $w\colon = |\nabla u|^2$. There exists a constant $C_1 = C_1(n, N, p, \Lambda)$ such that for each $\alpha \geq 0$ we have*

$$\begin{aligned}\frac{p-2+\alpha}{8}\int_\Omega \zeta^2 w^{\frac{p}{2}+\alpha-2}\,|\nabla w|^2\,dx + \frac{1}{2}\int_\Omega \zeta^2 w^{\frac{p}{2}+\alpha-1}|\nabla^2 u|^2\,dx \\ \leq \left(\frac{2(p-1)^2}{p-2+2\alpha}+\frac{1}{2}\right)\int_\Omega |\nabla\zeta|^2 w^{\frac{p}{2}+\alpha}\,dx + C_1(p+\alpha)\int_\Omega \zeta^2 w^{\frac{p}{2}+\alpha+1}\,dx\,,\end{aligned} \tag{2.1}$$

provided the right hand side is finite.

Proof. Differentiating both sides of (1.4) with respect to x_j, we see that for each test function ϕ_{ij} with compact support,

$$\int_\Omega \left[|\nabla u|^{p-2}\nabla\left(\frac{\partial u^i}{\partial x_j}\right) + \frac{\partial}{\partial x_j}\left(|\nabla u|^{p-2}\right)\nabla u^i\right]\cdot\nabla\phi_{ij}\,dx = -\int_\Omega \frac{\partial\phi_{ij}}{\partial x_j}G(x,u,\nabla u)\,dx\,. \tag{2.2}$$

Now, insert here $\phi_{ij} = \zeta^2|\nabla u|^{2\alpha}u^i_{x_j}$, where $\alpha \geq 0$ and $\zeta \in C_0^\infty(\Omega)$ is nonnegative,
Left hand side of (2.2). A simple but tedious computation leads to the following two equalities (indices i, j are summed):

$$\begin{aligned}\int_\Omega |\nabla u|^{p-2}\nabla\left(\frac{\partial u^i}{\partial x_j}\right)\cdot\nabla\phi_{ij}\,dx &= \int_\Omega \zeta^2 w^{\frac{p-2}{2}+\alpha}\left|\nabla\left(\frac{\partial u^i}{\partial x_j}\right)\right|^2\,dx \\ &\quad + \frac{\alpha}{2}\int_\Omega \zeta^2 w^{\frac{p-2}{2}+\alpha-1}|\nabla w|^2\,dx + \int_\Omega \zeta(\nabla\zeta\cdot\nabla w)w^{\frac{p-2}{2}+\alpha}\,dx \\ &=: I_1 + I_2 + I_3\,;\end{aligned} \tag{2.3}$$

$$\begin{aligned}&\int_\Omega \frac{\partial}{\partial x_j}\left(|\nabla u|^{p-2}\right)\nabla u^i\cdot\nabla\phi_{ij}\,dx \\ &= \frac{p-2}{4}\int_\Omega \zeta^2 w^{\frac{p-2}{2}+\alpha-1}|\nabla w|^2\,dx \\ &\quad + \frac{(p-2)\alpha}{2}\int_\Omega \zeta^2 w^{\frac{p-2}{2}+\alpha-2}\sum_i\left(\nabla w\cdot\nabla u^i\right)^2\,dx \\ &\quad + (p-2)\int_\Omega \zeta w^{\frac{p-2}{2}+\alpha-1}\sum_i(\nabla\zeta\cdot\nabla u^i)(\nabla w\cdot\nabla u^i)\,dx \\ &=: I_4 + I_5 + I_6\,.\end{aligned} \tag{2.4}$$

Using the familiar inequality $ab \leq \frac{\varepsilon^2 a^2}{2} + \frac{b^2}{2\varepsilon^2}$, we write

$$\begin{aligned}|I_3| + |I_6| &\leq (p-1)\int_\Omega \zeta|\nabla\zeta|\,|\nabla w|w^{\frac{p-2}{2}+\alpha}\,dx \\ &\leq \frac{(p-1)\varepsilon^2}{2}\int_\Omega \zeta^2 w^{\frac{p-2}{2}+\alpha-1}|\nabla w|^2\,dx + \frac{p-1}{2\varepsilon^2}\int_\Omega |\nabla\zeta|^2 w^{\frac{p}{2}+\alpha}\,dx\,.\end{aligned} \tag{2.5}$$

Choosing ε^2 so that $(p-1)\varepsilon^2/2 = (p-2+2\alpha)/8$, and combining (2.3), (2.4) and (2.5), we obtain finally

$$\begin{aligned}\text{left hand side of (2.2)} \geq \frac{p-2+2\alpha}{8} & \int_\Omega \zeta^2 w^{\frac{p-2}{2}+\alpha-1} |\nabla w|^2 \, dx \\ & + \int_\Omega \zeta^2 w^{\frac{p-2}{2}+\alpha} |\nabla u^i_{x_j}|^2 \, dx \\ & + \frac{(p-2)\alpha}{2} \int_\Omega \zeta^2 w^{\frac{p-2}{2}+\alpha-2} \sum_i \left(\nabla w \cdot \nabla u^i\right)^2 \, dx \\ & - \frac{2(p-1)^2}{p-2+2\alpha} \int_\Omega |\nabla\zeta|^2 w^{\frac{p}{2}+\alpha} \, dx \, .\end{aligned} \tag{2.6}$$

Right hand side of (2.2). Using the growth condition $|G(x,u,\nabla u)| \leq \Lambda |\nabla u|^p$, we write

$$\left| \int_\Omega \frac{\partial \phi_{ij}}{\partial x_j} G^i(x,u,\nabla u) \, dx \right| \leq C(J_1 + J_2 + J_3) \, , \tag{2.7}$$

where the constant $C = C(n, N, \Lambda)$ and

$$\begin{gathered} J_1 = \int_\Omega \zeta^2 w^{\frac{p}{2}+\alpha} \left| \nabla u^i_{x_j} \right| dx \, , \qquad J_2 = \alpha \int_\Omega \zeta^2 w^{\frac{p-1}{2}+\alpha} \left| \nabla w \right| \, dx \, , \\ J_3 = \int_\Omega \zeta |\nabla\zeta| w^{\frac{p+1}{2}+\alpha} \, dx \, . \end{gathered} \tag{2.8}$$

Set

$$J_0 \colon = \int_\Omega \zeta^2 w^{\frac{p}{2}+\alpha+1} \, dx \, .$$

To absorb all terms that contain second order derivatives of u, we apply Cauchy–Schwarz inequality in a familiar way and obtain

$$\begin{aligned} J_1 & \leq \frac{\varepsilon_1^2}{2} \int_\Omega \zeta^2 w^{\frac{p-2}{2}+\alpha} \left| \nabla u^i_{x_j} \right|^2 dx + \frac{1}{2\varepsilon_1^2} J_0 \, , \\ J_2 & \leq \frac{\alpha\varepsilon_2^2}{2} \int_\Omega \zeta^2 w^{\frac{p}{2}+\alpha-2} \left| \nabla w \right|^2 \, dx + \frac{\alpha}{2\varepsilon_2^2} J_0 \, . \end{aligned}$$

Finally,

$$J_3 \leq \frac{C}{2} J_0 + \frac{1}{2C} \int_\Omega |\nabla\zeta|^2 w^{\frac{p}{2}+\alpha} \, dx \, . \tag{2.9}$$

Making appropriate choices of $\varepsilon_1, \varepsilon_2 > 0$, we combine the estimates of J_1, J_2, J_3 with (2.6) and easily complete the proof of the lemma. □

3 Interpolation inequality

The following theorem provides the main new ingredient in our proof of boundedness of the gradient.

Theorem 3.1 *Let $\psi \in C_0^\infty(\mathbb{R}^n)$ be nonnegative, $s \geq 2$. There exists a constant $C = C(n)$, depending only on n, such that*

$$\int \psi^{s+2}|\nabla u|^{s+2}\,dx \leq Cs^2\|u\|_{BMO}^2\bigg\{\int \psi^{s+2}|\nabla u|^{s-2}|\nabla^2 u|^2\,dx + \|\nabla\psi\|_{L^\infty}^2 \int \psi^s|\nabla u|^s\,dx\bigg\} \tag{3.1}$$

for each function $u \in W^{2,1}_{\rm loc} \cap BMO$ for which the right hand side is finite.

The case $s = 2$ of this theorem, obtained by Littlewood–Paley theory, was applied by Y. Meyer and the first author in the proof of regularity of a certain class of Yang–Mills connections, see [20]. We do not know how to adapt the Fourier analytic proof from [20] to encompass the current situation, where a power of $|\nabla u|$ plays the role of a weight.

Since the argument below is based on Fefferman's duality theorem, let us recall that the Hardy space $\mathcal{H}^1(\mathbb{R}^n)$ consists precisely of those $g \in L^1(\mathbb{R}^n)$ for which

$$g_*\colon = \sup_{\varepsilon>0} |\varphi_\varepsilon * g| \in L^1(\mathbb{R}^n).$$

Here and below, $\varphi_\varepsilon(x) := \varepsilon^{-n}\varphi(x/\varepsilon)$ for a fixed $\varphi \in C_0^\infty(B(0,1))$ such that $\varphi \geq 0$ and $\int \varphi(y)\,dy = 1$. The definition does not depend on the choice of φ (see [8]); $\mathcal{H}^1(\mathbb{R}^n)$ is a Banach space with the norm $\|g\|_{\mathcal{H}^1} = \|g\|_{L^1} + \|g_*\|_{L^1}$, and its dual space $\big(\mathcal{H}^1(\mathbb{R}^n)\big)^* = BMO(\mathbb{R}^n)$; see [7], [8], or Stein's monograph [26, Chap. 4].

Proof of Theorem 3.1. The structure of the argument is quite simple and modelled on [29]. First, we apply a trick similar to that used e.g. by Coifman et al. [3] to prove that the Jacobian $\det Df$ of $f \in W^{1,n}(\mathbb{R}^n,\mathbb{R}^n)$ belongs to the Hardy space. That is, integration by parts and Sobolev inequality yield estimates below natural exponents. This is enough to conclude that $\operatorname{div}\big(|\nabla u|^s\nabla u\big)$ is not just integrable but also belongs to the Hardy space. (A cancellation condition, which was a must in [3], is replaced here by the existence of second order derivatives of u.) Fefferman's duality theorem closes the argument.

Here are the details.

Set $a\colon = s/2$. Integrating by parts, we write

$$\int_{\mathbb{R}^n} \psi^{2a+2}|\nabla u|^{2a+2}\,dx = \left|\int_{\mathbb{R}^n} u \operatorname{div} V\,dx\right|, \tag{3.2}$$

where $V := \psi^{2a+2}|\nabla u|^{2a}\nabla u$. We shall show that $g := \operatorname{div} V \in \mathcal{H}^1(\mathbb{R}^n)$, i.e., that

$$g_*\colon = \sup_{\varepsilon>0} |\varphi_\varepsilon * g| \in L^1(\mathbb{R}^n)\,. \tag{3.3}$$

Fix $x \in \mathbb{R}^n$ and $\varepsilon > 0$. We split $V = V_1 + V_2 + V_3$ and, accordingly, $g = g_1 + g_2 + g_3$, where $g_i = \operatorname{div} V_i$, setting

$$\begin{aligned} V_1 &= \psi^{a+1}|\nabla u|^{a+1}\Big(\psi^{a+1}|\nabla u|^{a-1}\nabla u - \fint_{B_\varepsilon} \psi^{a+1}|\nabla u|^{a-1}\nabla u \, dy\Big), \\ V_2 &= \psi^{a+1}|\nabla u|^{a+1}\Big(\fint_{B_\varepsilon} \psi^{a+1}|\nabla u|^{a-1}\nabla u \, dy - \psi \fint_{B_\varepsilon} \psi^{a}|\nabla u|^{a-1}\nabla u \, dy\Big), \\ V_3 &= \psi^{a+2}|\nabla u|^{a+1} \fint_{B_\varepsilon} \psi^{a}|\nabla u|^{a-1}\nabla u \, dy \end{aligned}$$

We have $|\varphi_\varepsilon * g| \le \sum_{i=1}^3 |\varphi_\varepsilon * g_i|$.

Step 1. Estimates of $g_{1,*} = \sup_{\varepsilon>0} |\varphi_\varepsilon * g_1|$. Set $f = \psi^{a+1}|\nabla u|^{a+1}$ and $h = \psi^{a+1}|\nabla u|^{a-1}\nabla u$. Integrating by parts, applying Hölder inequality with exponents $2n/(n+1)$ and $2n/(n-1)$, and then Sobolev inequality, we obtain

$$\begin{aligned} |g_1 * \varphi_\varepsilon| &\le \frac{C(n)}{\varepsilon} \fint_{B_\varepsilon} |f||h - h_{B_\varepsilon}| \, dy \\ &\le \frac{C(n)}{\varepsilon}\Big(\fint_{B_\varepsilon} |f|^{2n/(n+1)} \, dy\Big)^{(n+1)/2n} \Big(\fint_{B_\varepsilon} |h - h_{B_\varepsilon}|^{2n/(n-1)} \, dy\Big)^{(n-1)/2n} \\ &\le C(n)\Big(\fint_{B_\varepsilon} |f|^{2n/(n+1)} \, dy\Big)^{(n+1)/2n} \Big(\fint_{B_\varepsilon} |\nabla h|^{2n/(n+1)} \, dy\Big)^{(n+1)/2n}, \end{aligned} \tag{3.4}$$

Thus, upon taking supremum w.r.t. ε,

$$|g_{1,*}| \le C(n)\big(M|f|^{2n/(n+1)}\big)^{(n+1)/2n}\big(M|\nabla h|^{2n/(n+1)}\big)^{(n+1)/2n} \tag{3.5}$$

As $2n/(n+1) < 2$, Hardy–Littlewood maximal theorem yields now

$$\begin{aligned} \|g_{1,*}\|_{L^1} &\le C(n)\|f\|_{L^2}\|\nabla h\|_{L^2} \\ &= C(n)\Big(\int \psi^{2a+2}|\nabla u|^{2a+2} \, dx\Big)^{1/2}\Big(\int \big|\nabla(\psi^{a+1}|\nabla u|^{a-1}\nabla u)\big|^2 \, dx\Big)^{1/2}. \end{aligned} \tag{3.6}$$

Step 2. Estimates of $g_{2,*} = \sup_{\varepsilon>0} |\varphi_\varepsilon * g_2|$. By the mean value theorem,

$$\Big|\psi \fint_{B_\varepsilon} \psi^a|\nabla u|^{a-1}\nabla u \, dy - \fint_{B_\varepsilon} \psi^{a+1}|\nabla u|^{a-1}\nabla u \, dy\Big| \le 2\varepsilon\|\nabla\psi\|_{L^\infty} \fint_{B_\varepsilon} \psi^a|\nabla u|^a \, dy\,. \tag{3.7}$$

Integrating by parts and applying the above inequality, we obtain

$$|g_2 * \varphi_\varepsilon| \le \frac{C(n)}{\varepsilon}\varepsilon\|\nabla\psi\|_{L^\infty} \fint_{B_\varepsilon} \psi^a|\nabla u|^a \, dy \cdot \fint_{B_\varepsilon} \psi^{a+1}|\nabla u|^{a+1} \, dy,$$

whence

$$|g_{2,*}| \le C(n)\|\nabla\psi\|_{L^\infty} M(\psi^a|\nabla u|^a) M(\psi^{a+1}|\nabla u|^{a+1}),$$

and, by the maximal theorem,

$$\|g_{2,*}\|_{L^1} \le C(n)\|\nabla\psi\|_{L^\infty}\left(\int \psi^{2a}|\nabla u|^{2a}\,dx\right)^{1/2}\left(\int \psi^{2a+2}|\nabla u|^{2a+2}\,dx\right)^{1/2}. \tag{3.8}$$

Step 3. Estimates of $g_{3,*} = \sup_{\varepsilon>0}|\varphi_\varepsilon * g_3|$.

Here we do integrate the divergence by parts but rather estimate the convolution directly, writing

$$|\varphi_\varepsilon * g_3(x)| \le C(n) \fint_{B_\varepsilon} \psi^a|\nabla u|^a\,dy \fint_{B_\varepsilon}\left|\nabla\left(\psi^{a+2}|\nabla u|^{a+1}\right)\right|dy\,.$$

Computing the gradient in the second integral, we obtain

$$\begin{aligned}|g_{3,*}| &\le C(n)(a+2)\|\nabla\psi\|_{L^\infty}M(\psi^a|\nabla u|^a)M(\psi^{a+1}\nabla u|^{a+1})\\ &\quad + C(n)(a+1)M(\psi^a|\nabla u|^a)M(\psi^{a+2}|\nabla u|^a|\nabla^2 u|)\\ &=: F_1 + F_2\,.\end{aligned} \tag{3.9}$$

The term F_1 is in L^1, since both maximal functions are in L^2. Moreover, the L^1 norm of F_1 can be added to the right hand side of (3.8). One has just to increase the constant in (3.8) by a factor $\approx (a+2)$. To cope with F_2, note that

$$M(\psi^a|\nabla u|^a) \in L^{(2a+2)/a}, \qquad M(\psi^{a+2}|\nabla u|^a|\nabla^2 u|) \in L^{(2a+2)/(a+2)}\,. \tag{3.10}$$

This follows easily from Hölder inequality and Hardy–Littlewood maximal theorem. For $a \ge 1$ we have $(2a+2)/a \in [2,4]$ and the conjugate $(2a+2)/(a+2) \in [4/3, 2]$. Thus, the constants in two applications of the maximal theorem needed to obtain (3.10) were uniformly bounded by a factor $C = C(n)$.

Hence

$$\begin{aligned}\|F_2\|_{L_1} &\le C(n)(a+1)\|\psi^a|\nabla u|^a\|_{L^{(2a+2)/a}}\left\|\psi^{a+2}|\nabla u|^a|\nabla^2 u|\right\|_{L^{(2a+2)/(a+2)}}\\ &\le C(n)(a+1)\|\psi^a|\nabla u|^a\|_{L^{(2a+2)/a}}\left\|\psi|\nabla u|\right\|_{L^{2a+2}}\left\|\psi^{a+1}|\nabla u|^{a-1}|\nabla^2 u|\right\|_{L^2}\\ &= C(n)(a+1)\left(\int \psi^{2a+2}|\nabla u|^{2a+2}\,dx\right)^{1/2}\left(\int \psi^{2a+2}|\nabla u|^{2a-2}|\nabla^2 u|^2\,dx\right)^{1/2}.\end{aligned}$$

Adding the estimate of F_1, we obtain finally

$$\begin{aligned}\|g_{3,*}\|_{L^1} &\le (a+2)\cdot \text{the right hand side of (3.8)}\\ &\quad + C(n)(a+1)\left(\int \psi^{2a+2}|\nabla u|^{2a+2}\,dx\right)^{1/2}\left(\int \psi^{2a+2}|\nabla u|^{2a-2}|\nabla^2 u|^2\,dx\right)^{1/2}.\end{aligned} \tag{3.11}$$

It follows from (3.6), (3.8) and (3.11) that $g = \operatorname{div} V \in \mathcal{H}^1(\mathbb{R}^n)$. Thus, by Fefferman's duality theorem, (3.2) yields

$$\int_{\mathbb{R}^n} \psi^{2a+2}|\nabla u|^{2a+2}\,dx \le C(n)\|u\|_{BMO}\|g\|_{\mathcal{H}^1}\,.$$

Invoking again (3.6), (3.8) and (3.11) to estimate $\|g\|_{\mathcal{H}^1}$, cancelling the factor

$$\left(\int \psi^{2a+2}|\nabla u|^{2a+2}\,dx\right)^{1/2}$$

and squaring the result, we conclude the whole argument. □

4 Gradient bounds

As we already mentioned in the introduction, the strategy behind the proof of ε-regularity is quite simple: Theorem 3.1 is used to absorb the integral of $|\nabla u|^{p+2+2\alpha}$, which appears in the right hand side of Caccioppoli inequality (2.1), and then Moser iteration works. There are, however, some technical difficulties. First of all, this absorption trick can be applied only finitely many times. This is due to the fact that the constant in Theorem 3.1 blows up to ∞ as $p+2\alpha = s \to \infty$. Hence, we perform only finitely many steps of classical Moser iteration, absorbing the unpleasant term with $|\nabla u|^{p+2+2\alpha}$ as long as the BMO norm of u is small enough. This yields $\nabla u \in L^q$ for some large q, with appropriate bounds, and then it turns out that the gain of integrability given by the Sobolev imbedding is more important than the influence of the critical term in the Caccioppoli inequality. A modified version of Moser iteration shows then that $\nabla u \in L^\infty$, and gives an appropriate bound for all solutions with small scaled energy.

One has to control all the constants carefully, therefore we give all details, dividing the computation into two natural stages.

4.1 Initial L^q estimates

Let C_1 and C_2 denote the constants in, respectively, (2.1) and (3.1). We impose the following smallness condition: Assume that

$$C_1C_2(p+2\alpha)^3\|u\|^2_{BMO} \le \frac{1}{2} \tag{4.1}$$

for all $\alpha \in [0, \alpha_{\max}]$, where $\alpha_{\max} = \alpha_{\max}(n,p)$ is some sufficiently large number that shall be specified later.

Pick two nonnegative functions $\zeta, \psi \in C_0^\infty$ so that $\zeta^2 = \psi^{p+2\alpha+2}$, $\psi \equiv 1$ on a ball B_r, $\psi \equiv 0$ off a larger concentric ball B_R, and $|\nabla\psi| \le 2(R-r)^{-1}$. Next, write down the Caccioppoli inequality (2.1), and apply Theorem 3.1 (setting there $s = p+2\alpha$) to estimate the nasty term

$$C_1(p+\alpha)\int_\Omega \zeta^2|\nabla u|^{p+2\alpha+2}\,dx\,.$$

The smallness condition (4.1) allows to absorb the resulting integral of $\zeta^2|\nabla u|^{p+2\alpha-2}|\nabla^2 u|^2$ in the second term of the left hand side of the Caccioppoli inequality. This yields

$$\int_{B_r} w^{\frac{p}{2}+\alpha-2}|\nabla w|^2\,dx \le \frac{C_3(p+2\alpha)^2}{(R-r)^2}\int_{B_R} w^{\frac{p}{2}+\alpha}\,dx\,. \tag{4.2}$$

As before, $w \equiv |\nabla u|^2$. The constant C_3 depends only on n, N, p and Λ.

Set $\kappa = n/(n-2)$. From now on we assume that $r < R \leq 2r$. Applying the Sobolev inequality

$$\left(\int_{B_r} |f|^{2\kappa}\, dx\right)^{1/\kappa} \leq C(n) \int_{B_r} |\nabla f|^2\, dx + C(n)\, r^{-2} \int_{B_r} f^2\, dx \tag{4.3}$$

to $f :\, = w^{(p+2\alpha)/4}$, we obtain from (4.2) the following reverse Hölder inequality:

$$\left(\int_{B_r} |\nabla u|^{(p+2\alpha)\kappa}\, dx\right)^{1/\kappa} \leq \frac{C_4(p+2\alpha)^2}{(R-r)^2} \int_{B_R} |\nabla u|^{p+2\alpha}\, dx\, . \tag{4.4}$$

We shall now iterate this inequality, starting from $\alpha = 0$. We insert here $R = \varrho_j$, $r = \varrho_{j+1}$, $\alpha \equiv \alpha_j$, where the sequences (ϱ_j) and (α_j) are defined by

$$\begin{aligned} \varrho_j &= \varrho\left(\frac{1}{2} + \frac{1}{2^j}\right), \qquad j = 1, 2, \ldots, \\ \alpha_j &= \frac{p}{2}\left(\kappa^{j-1} - 1\right), \qquad j = 1, 2, \ldots \end{aligned} \tag{4.5}$$

We also set $q_j :\, = p + 2\alpha_j$ and

$$I(r,q) = \left(r^q \fint_{B_r} |\nabla u|^q\, dx\right)^q . \tag{4.6}$$

Using this notation, we rewrite (4.4) as

$$\begin{aligned} I(\varrho_{j+1}, q_{j+1}) &\leq \left(C_3(n,p) q_j^2\right)^{1/q_j} \left(\frac{\varrho_{j+1}}{\varrho_j - \varrho_{j+1}}\right)^{2/q_j} I(\varrho_j, q_j) \\ &\leq C_4\big(n,p\big)^{j/\kappa^j} I(\varrho_j, q_j)\, . \end{aligned} \tag{4.7}$$

We iterate (4.7) as long as the smallness condition (4.1) is satisfied, to obtain

$$\left(\left(\frac{\varrho}{2}\right)^q \fint_{B_{\varrho/2}} |\nabla u|^q\, dx\right)^{1/q} \leq C_5 \left(\varrho^p \fint_{B_\varrho} |\nabla u|^p\, dx\right)^{1/p} \qquad \text{for all } q \leq q_{\max}. \tag{4.8}$$

One can take $C_5 = C_5(n, N, p, \Lambda)$ given by

$$\log C_5 = \log C_4 \sum_{j=1}^{\infty} \frac{j}{\kappa^j}\, . \tag{4.9}$$

This proves Theorem 1.1; one has just to adjust $\alpha_{\max}$ (and hence ε_0) to obtain q large enough.

For the remaining part of computations that are necessary to prove Theorem 1.2, it is convenient to choose $q_{\max} = p + 6n$, $\alpha_{\max} = 3n$. Obviously, no matter how $\alpha_{\max}$ is chosen, the smallness condition (4.1) fails to be satisfied for some sufficiently large α unless $u \equiv \text{const}$. Therefore, to obtain uniform L^q gradient bounds for large q (including $q = \infty$) we must change classical Moser's argument slightly. This is done in the next subsection.

4.2 Boundedness of the gradient

We shall show now that if the scaled energy of u is small, i.e.

$$\left(\varrho^p \fint_{B_\varrho} |\nabla u|^p \, dx\right)^{1/p} < \varepsilon_1(n,p,N,\Lambda)\,, \tag{4.10}$$

where $\varepsilon_1(n,p,N,\Lambda)$ shall be specified later on, then $|\nabla u| \in L^\infty(B_{\varrho/4})$. All numbered constants C_i below depend only on n, N, p and Λ. We fix a ball B_ϱ for which *both* smallness conditions (4.1) and (4.10) hold.

Let us return to the Caccioppoli inequality (2.1). Dropping the second term in the left hand side of (2.1), and choosing a standard cutoff function ζ, we obtain

$$\begin{aligned} \frac{p-2+\beta}{8} \int_{B_r} w^{\frac{p}{2}+\beta-2} |\nabla w|^2 \, dx \le \frac{C_6}{(R-r)^2} \int_{B_R} w^{\frac{p}{2}+\beta} \, dx \\ + C_6(p+2\beta) \int_{B_R} w^{\frac{p}{2}+\beta+1} \, dx\,. \end{aligned} \tag{4.11}$$

As before, $w = |\nabla u|^2$, and $B_r \subset B_R$ denote two concentric balls. We shall use this inequality only for $\beta \ge 2n$ and for $r < R < 2r$, assuming always that $B_R \subset B_{\varrho/2}$. Combining (4.11) with Sobolev inequality (4.3), we get

$$\begin{aligned} \left(\int_{B_r} |\nabla u|^{(p+2\beta)\kappa}\right)^{1/\kappa} \le C_7(p+2\beta)^2 \Bigg\{ \frac{1}{(R-r)^2} \int_{B_R} |\nabla u|^{p+2\beta} \, dx \\ + \int_{B_R} |\nabla u|^{p+2\beta+2} \, dx \Bigg\}\,. \end{aligned} \tag{4.12}$$

Note that for β large enough, in particular for all $\beta \ge 2n$, we have $(p+2\beta)\kappa > p+2\beta+2$. Thus, for such β inequality (4.12) yields indeed higher integrability of ∇u.

Multiplying both sides of (4.12) by $r^{p+2\beta+2-n}$ and applying Hölder inequality to get two equal integrands in the right hand side of (4.12), we easily see that

$$\begin{aligned} I\big(r,(p+2\beta)\kappa\big)^{p+2\beta} \le C_8(p+2\beta)^2 \Bigg\{ \frac{r^2}{(R-r)^2} I\big(R,p+2\beta+2\big)^{p+2\beta} \\ + I\big(R,p+2\beta+2\big)^{p+2\beta+2} \Bigg\}\,, \end{aligned} \tag{4.13}$$

where $I(\varrho,q)$ is defined by (4.6).

We shall iterate (4.13) for a sequence of nested concentric balls and for a sufficiently chosen sequence of exponents $s = p+2\beta+2 \to \infty$. Namely, let $r = r_{j+1}$ and $R = r_j$, where

$$r_j = \frac{\varrho}{2}\left(\frac{1}{2} + \frac{1}{2^j}\right) \qquad \text{for } j = 1,2,\ldots,$$

and set $\beta := \beta_j$, $s_j := p+2\beta_j+2$, where the sequence (β_j) is defined recursively by the formulae

$$\beta_1 = 2n \tag{4.14}$$

$$\beta_{j+1} = \frac{p}{2}(\kappa - 1) + \beta_j\kappa - 1, \qquad j = 1, 2, \dots \tag{4.15}$$

so that for each j we have

$$s_{j+1} = p + 2\beta_{j+1} + 2 = (p + 2\beta_j)\kappa\,.$$

It is easy to check that

$$\beta_{j+1} = \frac{p}{2}(\kappa^j - 1) + \beta_1\kappa^j - \left(1 + \kappa + \cdots + \kappa^{j-1}\right).$$

Thus

$$(p + 3n)\kappa^j < p + 2\beta_{j+1} < (p + 4n)\kappa^j \qquad \text{for all } j. \tag{4.16}$$

Using these observations and (4.13), it is easy to see that *as long as* the treshold condition

$$I(r_j, s_j) \le 1 \tag{4.17}$$

is satisfied, we have

$$\begin{aligned} I(r_{j+1}, s_{j+1}) &\le \left(C_8(p + 2\beta_j)^2\right)^{1/(p+2\beta_j)} \left(8^j + 1\right)^{1/(p+2\beta_j)} I(r_j, s_j) \\ &\le C_9^{j/\kappa^j} I(r_j, s_j)\,. \end{aligned} \tag{4.18}$$

Now, for $j = 1$ we have $r_1 = \varrho/2$ and $s_1 = p + 2\beta_1 + 2 < q_{\max} = p + 6n$. Hence, for $j = 1$ (4.17) clearly holds, in light of (4.8) and (4.10); one just has to choose ε_1 not greater than $1/C_5$.

By induction, *if* (4.17) is satisfied for $j = 1, 2, \dots, k$, then

$$\begin{aligned} I(r_{k+1}, s_{k+1}) &\le C_9^{\sum_{j=1}^k j/\kappa^j} I(r_1, s_1) \\ &\le C_{10}C_5\left(\varrho^p \fint_{B_\varrho} |\nabla u|^p\, dx\right)^{1/p}, \end{aligned} \tag{4.19}$$

where $\log C_{10}/\log C_9 = \sum_{j=1}^\infty j/\kappa^j$. Thus, choosing $\varepsilon_1(n, p, N, \Lambda) = (C_{10}C_5)^{-1} > 0$ in (4.10), we obtain the treshold condition (4.17) for each j. Then, upon passing to the limit $k \to \infty$ in (4.19), we conclude that

$$\operatorname*{ess\,max}_{B_{\varrho/4}} |\nabla u| \le \frac{C}{\varrho}\left(\varrho^p \fint_{B_\varrho} |\nabla u|^p\, dx\right)^{1/p}. \tag{4.20}$$

This completes the proof of Theorem 1.2.

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