The singular set of J-holomorphic maps into projective algebraic varieties

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Abstract : We prove that the singular set of any $J-$holomorphic map from a 4-dimensional almost complex manifold into a projective algebraic variety is made of isolated points .

I Introduction

Let (M^4, J) be a smooth almost complex manifold of dimension 4. A map u in $W^{1,2}_{loc}(M^4, \mathbb{C}P^n)$ is said to be weakly J-harmonic if it preserves the almost-complex structures in the following sense :

$$\begin{array}{l} \text{for a.e. } x \in M^4 \qquad \forall X \in T_x M^4 \\ du_x \; JX = j_0 \; du_x X \end{array} \tag{I.1}$$

where j_0 is the standard integrable complex structure on $\mathbb{C}P^n$.

A map u in $W^{1,2}_{loc}(M^4, \mathbb{C}P^n)$ is said to be *locally approximable* if for any ball $\overline{B}^4 \subset M^4$, there exists $u_n \in C^\infty(M^4, \mathbb{C}P^n)$ such that

$$u_n \longrightarrow u \qquad \text{strongly in } W^{1,2}(B^4, \mathbb{C}P^n)$$

As explained in [BCDH], u is locally approximable if and only if

$$d(u^*\omega) = 0 \qquad \text{in } \mathcal{D}'(M^4) \tag{I.2}$$

where ω is any given $2-$form generating $H^2(\mathbb{C}P^n)$. Notice that condition (I.2) does not necessarily ensure the global approximability of u on the whole of M^4 (see recent developpements on these questions of global approximability for maps in Sobolev Spaces between manifolds in [HL]).

In this paper we will focus on smoothness properties of $J-$holomorphic maps which are locally approximable taking values into the complex projective spaces (Maps solving (I.1) and (I.2)). As the integrable case suggests, looking for instance at the map

$$\begin{array}{rcl} h\ :\ \mathbb{C}^2 & \longrightarrow & \mathbb{C}P^1 \\ (z_1,z_2) & \longrightarrow & [z_1,z_2] \end{array} \tag{I.3}$$

which is holomorphic, singular at the origin and which solves (I.2), one has to expect locally approximable $J-$holomorphic maps to have singularities. Our main result is

Theorem I.1 *Any locally approximable J-holomorphic map from an almost complex manifold (M^4,J) into an algebraic variety $N\subset\mathbb{C}P^n$ has isolated point singularities.*

Our bound of the size of the singular set, in view of the example (I.3), is clearly optimal. It is reasonable to expect this result to be extendable to any almost complex closed target. We also conjecture that for any dimension $m\ge 2$ and any given almost complex domain (M^{2m},J), any locally approximable $J-$holomorphic maps u from (M^{2m},J) into $\mathbb{C}P^n$ (or even into any almost complex closed manifold) have a closed singular set $\mathrm{Sing}\,u$ verifying

$$\mathcal{H}^{2m-4}(\mathrm{Sing}\,u\cap K)<+\infty$$

for any compact subset K of M^{2m}. Such results were established in the case where J, on M^{2m}, is integrable in [Ki] and [HS]. These works made an intensive use of the complex variables which have no equivalent in the non-integrable case.

Our result is local. We will then restrict to an arbitrary 4 dimensional ball $B^4\subset M^4$ on which J can be tamed : there exists a symplectic 2-form Ω such that for every $x\in B^4$ and every $X,Y\in T_xB^4$ we have $\Omega(X,JX)>0$ and $\Omega(JX,JY)=\Omega(X,Y)$ (the local existence of such an Ω is established in Lemma A.1). Let g be the Riemannian metric compatible with (Ω,J)

$$g_{\Omega,J}(X,Y)=\Omega(X,JY) \tag{I.4}$$

(J_x becomes a skew self adjoint isometry of T_xM^4 for g_x). At each given singular point we also have the following "uniqueness property of the tangent cone" :

Theorem I.2 *Let u be a $J-$holomorphic map from an almost complex manifold (B^4, J) into a projective algebraic variety N. Assume there exists a symplectic form on B^4 taming J. Let x_0 be a singular point for u and denote $u_{r,x_0}(x) = u(rx + x_0)$ the blow-up of u at x_0 for a given normal coordinate system about x_0 (for $g_{J,\Omega}$) that makes the J structure coincide at x_0 with the standard complex structure J_0 on $\mathbb{R}^4 \simeq \mathbb{C} \oplus \mathbb{C}$, then, there exists an holomorphic map ϕ from $\mathbb{C}P^1$ into N such that*

$$u_{r,x_0} \rightharpoonup u_\infty = \phi \circ h \qquad \text{weakly in } W^{1,2}(B^4_1) \tag{I.5}$$

where h is given by (I.3). Moreover, in B^4_1, the following convergence for Radon measures holds :

$$|\nabla u_{r,x_0}|^2 \ dx \rightharpoonup |\nabla u_\infty|^2 \ dx + 8\pi \sum_{k=1}^{q} N_k \ \mathcal{H}^2 \llcorner D_k \tag{I.6}$$

where dx is the Lebesgue measure on B^4_1, N_k are positive integers, D_k are $J_{x_0}-$holomorphic 2-disks and $\mathcal{H}^2 \llcorner D_k$ denotes the $2-$dimensional Hausdorff measure restricted to D_k.

As explained in part II of the paper, any locally approximable $J-$holomorphic map from (B^4, J, Ω) into $\mathbb{C}P^n$ is a stationary harmonic map from $(M^4, g_{J,\Omega})$ into $(\mathbb{C}P^n, g_0)$ where g_0 is the Fubini Study metric on $\mathbb{C}P^n$. Therefore our results enter in the family of regularity results for stationary harmonic maps. It is known from C.Evans, [Ev], that any stationary harmonic map u from a 4-dimensional domain (B^4, g) have a closed singular set whose 2-dimensional Hausdorff measure is 0 and which coincides with the set of points x_0 verifying

$$\lim_{r\to 0} \frac{1}{r^2} \int_{B_r(x_0)} |\nabla u|^2 \geq \varepsilon_0 > 0$$

where ε_0 is a positive constant depending only on $g_{J,\Omega}$ on B^4 and $B_r(x_0)$ is the geodesic ball of center x_0 and radius r. In the particular case where u is a minimizing harmonic map, it was known from R.Schoen and K.Uhlenbeck [SU] that the Hausdorff dimension of the singular set of u in M^4 is 1. The particular feature of locally approximable $J-$holomorphic maps is that they are minimizing but only among maps which are also locally approximable by smooth ones (see proposition II.3) whereas one may find sequences of smooth holomorphic maps u_n from $\mathbb{C}^2$ into $\mathbb{C}P^1$ such that

$$\frac{\displaystyle\inf_{v=u_n \text{ on } \partial B_1^4}\int_{B_1^4}|\nabla v|^2}{\displaystyle\int_{B_1^4}|\nabla u_n|^2}\longrightarrow 0$$

by producing maps v having topological singularities : i.e. verifying $d(v^*\omega)\neq 0$ in $\mathcal{D}'(B_1^4)$. This lack of being minimizing among all maps in $W^{1,2}$ but only among the local strong closure of C^∞ in $W^{1,2}$ permits the bubbling phenomenon described by F.H.Lin in [Li] and [Li2] to arise while dilating a $J-$holomorphic map about a singular point. This makes Federer dimension reduction argument used by R.Schoen and K.Uhlenbeck in [SU] break down in our situation. This is one of the reason that makes the regularity theory for $J-$holomorphic maps particularly interresting.

Our approach is based on the following observation. Preimages of regular points of locally approximable $J-$holomorphic maps from (M^4, J) into $\mathbb{C}P^1$ are rectifiable $J-$holomorphic cycles. The major part of the paper is devoted to the proof of the fact that these rectifiable $J-$holomorphic cycles are indeed $J-$holomorphic curves (almost complex submanifold of M^4, smooth away from finitely many branch points). Again this result was known, since the works of J.R. King [Ki] and R.Harvey, B.Shiffman [HS] in the particular case where J is assumed to be integrable. A proof of this result, in the case of general almost-complex structures J, for $J-$holomorphic cycles arising as the concentration set of solutions to Seiberg-Witten equations occupied the main part of the paper SW $\Longrightarrow$ Gr in [Ta]. Unfortunately we discovered some missing argument in this proof (see the remark III.1 below) and a substantial work has to be carried out to fill this gap. This is also one of the justifications of the present paper in which we give a self-contained and complete proof of the regularity of rectifiable $J-$holomorphic 2-cycles which are preimages of locally approximable $J-$holomorphic maps into $\mathbb{C}P^1$.

II Preliminaries

Recall first that a map u from (B^4, g) into $\mathbb{C}P^n$ (that we assumed to be isometrically embedded in some Euclidean space $\mathbb{R}^k$) is weakly harmonic if for any smooth map ξ from B^4 into $\mathbb{R}^k$

$$\frac{d}{dt}\left[\int_{B^4} |\nabla \pi_{\mathbb{C}P^n}(u+t\xi)|^2\right]_{|t=0} = 0 \tag{II.7}$$

where $\pi_{\mathbb{C}P^n}$ is the orthogonal projection of a neighborhood of $\mathbb{C}P^n$ in $\mathbb{R}^k$ into $\mathbb{C}P^n$.

Proposition II.1 *Let Ω be a symplectic form on B^4 that tames an almost complex structure J on B^4 (see Lemma A.1). Then, any weakly $J-$holomorphic map u into $\mathbb{C}P^n$ (solving (I.1)) is weakly harmonic from $(B^4, g_{J,\Omega})$ into $\mathbb{C}P^n$ and solves*

$$\Omega \wedge u^*\omega_{\mathbb{C}P^n} = \frac{|\nabla u|^2}{2}\frac{\Omega^2}{2!} \tag{II.8}$$

where $\omega_{\mathbb{C}P^n}$ is the standard Kähler form on $\mathbb{C}P^n$ (normalized so that $|\omega_{\mathbb{C}P^n}| = 1$ for the Fubini-Study metric - $|\ \cdot\ |$ denotes the comass norm).

Proof of proposition II.1 : From now on in the remaining of the paper, we consider $\mathbb{C}P^n$ through an isometrical embeddeding into some Euclidean space $\mathbb{R}^k$. Let $x \in B^4$ such that $du_x \neq 0$ and let $(\varepsilon^*_{2i-1},\ \varepsilon^*_{2i} = j_0\varepsilon^*_{2i-1})_{i=1\cdots n}$ be an orthonormal basis of $T^*_{u(x)}\mathbb{C}P^n$ (recall that j_0 is the standard complex structure on $\mathbb{C}P^n$). We have, using the $J-$holomorphic nature of u

$$u^*\omega_{\mathbb{C}P^n} = \sum_{i=1}^n u^*(\varepsilon^*_{2i-1} \wedge j_0\varepsilon^*_{2i-1}) = \sum_{i=1}^n u^*\varepsilon^*_{2i-1} \wedge Ju^*\varepsilon^*_{2i-1}$$

Therefore $\Omega \wedge u^*\omega_{\mathbb{C}P^n} = \sum_{i=1}^n \Omega \wedge u^*\varepsilon^*_{2i-1} \wedge Ju^*\varepsilon^*_{2i-1}$ and using the fact that for any covector ξ of T^*M^4 we have $\Omega \wedge \xi \wedge J\xi = |\xi|^2\frac{\Omega^2}{2!}$, we deduce from the previous identities

$$\Omega \wedge u^*\omega_{\mathbb{C}P^n} = \sum_{i=1}^n |u^*\varepsilon^*_{2i-1}|^2\ \frac{\Omega^2}{2!} = \frac{|\nabla u|^2}{2}\ \frac{\Omega^2}{2!}$$

Next we will deduce from (II.8) that u is weakly harmonic.

For any given map $v \in W^{1,2}(B^4, \mathbb{C}P^n)$, at a given point $x \in B^4$ we consider an orthonormal basis of T^*B^4 of the form $e_1^*, e_2^* = Je_1^*, e_3^*, e_4^* = Je_3^*$. In such a basis the almost Kähler form writes $\Omega = e_1^* \wedge e_2^* + e_3^* \wedge e_4^*$ we have

$$
\begin{aligned}
&*\Omega \wedge v^*\omega_{\mathbb{C}P^n} = *\sum_{i=1}^n e_1^* \wedge e_2^* \wedge v^*\varepsilon_{2i-1}^* \wedge v^*\varepsilon_{2i}^* + *\sum_{i=1}^n e_3^* \wedge e_4^* \wedge v^*\varepsilon_{2i-1}^* \wedge v^*\varepsilon_{2i}^* \\
&\le \sum_{i=1}^n \left| < e_3; v^*\varepsilon_{2i-1}^* > < e_4; v^*\varepsilon_{2i}^* > - < e_4; v^*\varepsilon_{2i-1}^* > < e_3; v^*\varepsilon_{2i}^* > \right| \\
&+ \sum_{i=1}^n \left| < e_1; v^*\varepsilon_{2i-1}^* > < e_2; v^*\varepsilon_{2i}^* > - < e_2; v^*\varepsilon_{2i-1}^* > < e_1; v^*\varepsilon_{2i}^* > \right| \\
&\le \frac{1}{2}\sum_{i=1}^n \sum_{k=1}^4 | < e_k; v^*\varepsilon_{2i-1}^* > |^2 + \frac{1}{2}\sum_{i=1}^n \sum_{k=1}^4 | < e_k; v^*\varepsilon_{2i}^* > |^2 \le \frac{|\nabla v|^2}{2}
\end{aligned}
\tag{II.9}
$$

Therefore we deduce that for any $\xi \in C_0^\infty(B^4)$ we have

$$
\int_{B^4} |\nabla \pi_{\mathbb{C}P^n} \circ (u + t\xi)|^2 \ge 2 \int_{B^4} \Omega \wedge (u + t\xi)^* \pi_{\mathbb{C}P^n}^* \omega_{\mathbb{C}P^n} \tag{II.10}
$$

with equality for $t = 0$ as stated in (II.8). For a.e. $x \in B^4$ we compute

$$
\frac{d}{dt}(u + t\xi)^* \pi_{\mathbb{C}P^n}^* \omega_{\mathbb{C}P^n} = d(u^* \iota_\xi(\pi_{\mathbb{C}P^n}^* \omega_{\mathbb{C}P^n})) + u^* \iota_\xi d(\pi_{\mathbb{C}P^n}^* \omega_{\mathbb{C}P^n}) = d\alpha_\xi^u
$$

where $\alpha_\xi^u = u^* \iota_\xi(\pi_{\mathbb{C}P^n}^* \omega_{\mathbb{C}P^n})$($\iota_\xi$ denotes the interior product by the vector $\xi(x) \in \mathbb{R}^k$). Observe that $\text{Supp}\alpha_\xi^u \subset B^4$. Therefore we deduce that for any $u \in W^{1,2}(B^4, \mathbb{C}P^n)$ and any $\xi \in C_0^\infty(B^4)$

$$
\frac{d}{dt} \int_{B^4} \Omega \wedge (u + t\xi)^* \pi_{\mathbb{C}P^n}^* \omega_{\mathbb{C}P^n} = 0
$$

In other words, $\int \Omega \wedge u^* \omega_{\mathbb{C}P^n}$ is a null Lagrangian. Since equality holds in (II.10) for any u being $J-$holomorphic, we deduce (II.7) and proposition II.1 is proved. ∎

Proposition II.2 *Any locally approximable $J-$holomorphic map from an almost complex 4-manifold (M^4, J) into $\mathbb{C}P^n$ is a stationary harmonic map.*

Proof of proposition II.2 : Recall that, by definition, u is stationary harmonic if u is weakly harmonic (solves (II.7)) and u is also a critical point of the Dirichlet Energy for perturbations in the domain B^4 :

$$\forall X \in C_0^\infty(B^4) \qquad \frac{d}{dt}\int_{B^4} |\nabla u \circ (id + tX)|^2 = 0 \tag{II.11}$$

Denote $\Psi_t = id + tX$. From (II.9), we have that for any $t \geq 0$

$$\int_{B^4} |\nabla(u \circ \Psi_t)|^2 \geq 2\int_{B^4} \Omega \wedge \Psi_t^* u^* \omega_{\mathbb{C}P^n} \tag{II.12}$$

with equality for $t = 0$ since u is $J-$holomorphic. We have, for a.e. $x \in B^4$,

$$\frac{d}{dt}\Psi_t^* u^* \omega_{\mathbb{C}P^1} = \mathcal{L}_X u^* \omega_{\mathbb{C}P^n} = d(\iota_X u^* \omega_{\mathbb{C}P^n}) + \iota_X d(u^* \omega_{\mathbb{C}P^n})$$

Using (I.2) we deduce that

$$\frac{d}{dt}\Psi_t^* u^* \omega_{\mathbb{C}P^n} = d\beta_X^u$$

where $\beta_X^u = \iota_X u^* \omega_{\mathbb{C}P^n} \in C_0^\infty(\wedge^1 B^4)$. Integrating by parts, we deduce that, for any u in $W^{1,2}$ which is locally approximable by smooth maps and $X \in C_0^\infty(B^4)$,

$$\frac{d}{dt}\int_{B^4} \Omega \wedge \Psi_t^* u^* \omega_{\mathbb{C}P^n} = 0$$

Again we have a null Lagrangian for perturbations of the domain and we deduce that (II.11) holds for any u for which (II.12) is an equality at $t = 0$. Proposition II.2 is then proved. ■

In fact we can even improve the previous proposition with the following result.

Proposition II.3 *A locally approximable $J-$holomorphic map u from (M^4, J) into $\mathbb{C}P^n$ minimizes the energy among the maps which are locally approximable in $W^{1,2}(M^4, \mathbb{C}P^n)$. In other words given any ball $\overline{B^4}$ in M^4 and any map $v \in W^{1,2}(M^4, \mathbb{C}P^n)$ such that $d(v^* \omega_{\mathbb{C}P^n}) = 0$ and verifying $v = u$ on $M^4 \setminus B^4$, then we have*

$$\int_{B^4} |\nabla u|^2 \leq \int_{B^4} |\nabla v|^2$$

Remark II.1 *This minimality property does not hold anymore while removing the approximability property for v. Indeed if $J-$holomorphic maps would be absolute minimizers, from [SU], we could deduce that any weak convergence of $J-$holomorphic maps in $W^{1,2}$ is in fact a strong one ! Such a property can easily be disproved by extending to $\mathbb{C}\oplus\mathbb{C}$ holomorphic maps from $\mathbb{C}$ into $\mathbb{C}P^1$ that shrink to a point.*

Proof of proposition II.3. Given a ball $\overline{B}^4$ and $v\in W^{1,2}(M^4,\mathbb{C}P^n)$ such that $u=v$ on $M^4\setminus B^4$. Take a metric on M^4 so that the ball B^4 we consider is a radius 1 ball (to fix ideas). First of all we claim that, given any $\delta>0$,(small enough, so that the ball $B^4_{1+\delta}$ retracts to B^4_1 which is diffeomorphic to the unit ball in $\mathbb{R}^4$) there exists α_u and α_v in $W^{2,2}(\wedge^1 B^4_{1+\delta})$ such that

$$
\begin{aligned}
&u^*\omega_{\mathbb{C}P^n}=d\alpha_u \quad v^*\omega_{\mathbb{C}P^n}=d\alpha_v \quad \text{ on } B^4_{1+\delta}\\
&\text{and such that } \alpha_v=\alpha_u \quad \text{ on } B^4_{1+\delta}\setminus B^4_{1+\delta/2}
\end{aligned}
\tag{II.13}
$$

First of all, by standard elliptic theory we can find $\tilde{\alpha}_u$ and $\tilde{\alpha}_v$ in $W^{2,2}(\wedge^1 B^4_{1+\delta})$ such that $u^*\omega_{\mathbb{C}P^n}=d\tilde{\alpha}_u$ and $v^*\omega_{\mathbb{C}P^n}=d\tilde{\alpha}_v$. Let χ be a function in $C^\infty(M^4)$ equal to 1 on $M^4\setminus B^4_{1+\delta/2}$ and equal to 0 in B^4_1. Since $u^*\omega_{\mathbb{C}P^n}-v^*\omega_{\mathbb{C}P^n}=d(\tilde{\alpha}_u-\tilde{\alpha}_v)=0$ on $B^4_{1+\delta}\setminus B^4_1$, there exists a function $\gamma\in W^{3,2}(B^4_{1+\delta}\setminus B^4_1)$ such that $\tilde{\alpha}_u-\tilde{\alpha}_v=d\gamma$ on $B^4_{1+\delta/2}\setminus B^4_1$. We choose then $\alpha_u:=\tilde{\alpha}_u$ and $\alpha_v:=\tilde{\alpha}_v+d(\chi\gamma)$. These two forms solve (II.13). We deduce in particular, since Ω is closed, that

$$
\int_{B^4_{1+\delta}}\Omega\wedge u^*\omega_{\mathbb{C}P^n}=\int_{\partial B^4_{1+\delta}}\Omega\wedge\alpha_u=\int_{\partial B^4_{1+\delta}}\Omega\wedge\alpha_v=\int_{B^4_{1+\delta}}\Omega\wedge v^*\omega_{\mathbb{C}P^n} \tag{II.14}
$$

Combining now (II.8), (II.9) and (II.14) we obtain

$$
\int_{B^4_{1+\delta}}\frac{|\nabla u|^2}{2}=\int_{B^4_{1+\delta}}\Omega\wedge u^*\omega_{\mathbb{C}P^n}=\int_{B^4_{1+\delta}}\Omega\wedge v^*\omega_{\mathbb{C}P^n}\le\int_{B^4_{1+\delta}}\frac{|\nabla v|^2}{2}
$$

which is the desired result and proposition II.3 is proved. ∎

Lemma II.1 *(Monotonicity Formula) Let u be a stationary harmonic map from (M^4,g) into $\mathbb{C}P^m$, then for any x_0 in M there exists $r_0>0$ and $a>0$*

(depending only on g in the geodesic ball $B_1(x_0)$) such that for any $\sigma < r < 1/2$ we have

$$\begin{aligned} &\frac{e^{ar^2}}{r^2}\int_{B_r(x_0)}|\nabla u|^2-\frac{e^{a\sigma^2}}{\sigma^2}\int_{B_\sigma(x_0)}|\nabla u|^2\\ &\geq 2\int_{B_r(x_0)\setminus B_\sigma(x_0)}\frac{e^{as^2}}{s^2}\left|\frac{\partial u}{\partial r}\right|^2 \end{aligned} \tag{II.15}$$

where $\frac{\partial}{\partial r}=\sum_{i=1}^4 x_i\frac{\partial}{\partial x_i}$ and x_i are normal coordinates about x_0.

Proof of lemma II.1 : The proof of the monotonicity formula for stationary harmonic maps is fairly standard. From the stationarity property (II.11) we deduce the stationarity equation : for any $X\in C_0^\infty(M^4)$

$$\int_{M^4} -div\, X\ |\nabla u|^2\ dx + 2\sum_{k=1}^4 < du;[X,e_k]>< du; e_k >=0 \tag{II.16}$$

e_k is an arbitrary orthonormal basis. Choose normal coordinates centered at x_0 in a geodesic ball $B_{r_0}(x_0)$: we have $x_1 ... x_4$ such that $x_i(x_0)=0$,

$$|g_{ij}-\delta_{ij}|\leq C\sum_i |x_i|^2 \quad ,$$

and

$$|dg_{ij}|\leq C\sqrt{\sum_i |x_i|^2}$$

Define $X=\xi(r)r\frac{\partial}{\partial r}$ for any function $\xi\in C_0^\infty(B^4_{r_0})$. We have, since the x_i are normal coordinates, $\nabla_{\frac{\partial}{\partial r}}\frac{\partial}{\partial r}=0$ (where ∇ is here the levi-Civita connection on M^4). Let e_i be an orthonormal frame on $B^4_{r_0}(x_0)$ chosen such that $e_1=\frac{\partial}{\partial r}$ (recall that S^3 is parallelizable). We have

$$\nabla_{e_1}X=\nabla_{\frac{\partial}{\partial r}}X=(r\xi'+\xi)\frac{\partial}{\partial r}$$

and for $i\geq 2$

$$\nabla_{e_i}X=\xi\sum_{j=1}^4 b_{ij}e_j$$

where $|b_{ij}-\delta_{ij}| = O(r^2)$. Applying (II.16), using that $[X, e_i] = \nabla_X e_i - \nabla_{e_i} X$, after some computations we obtain

$$\int_{B^4_{r_0}(x_0)} |du|^4 (r\xi' + 2\xi) + r^2 \xi Q(du) = 2 \int_{B^4_{r_0}(x_0)} r\ \xi' \left|\frac{\partial u}{\partial r}\right|^2 \tag{II.17}$$

where $Q(du) \leq a|du|^2$. We choose, for any $\tau > 0$ and $\varepsilon > 0$, $\xi(r) = \xi_\tau(r) = \eta(r/\tau)$ where $\eta \in C_0^\infty(\mathbb{R})$ solves $\eta = 1$ on $[0,1]$, $\eta = 0$ on $[1+\varepsilon, 0]$ and $\eta' \leq 0$. We have $\tau \frac{\partial}{\partial \tau}(\xi_\tau(r)) = -r\xi'_\tau$ and (II.17) becomes

$$\begin{aligned} &\tau \frac{\partial}{\partial \tau}\left(\int_{B^4} \xi_\tau |du|^2\right) - 2\int_{B^4} \xi_\tau |du|^2 - \int_{B^4} \xi_\tau r^2 Q(du) \\ &= 2\tau \frac{\partial}{\partial \tau} \int_{B^4} \xi_\tau \left|\frac{\partial u}{\partial r}\right|^2 \end{aligned}$$

integrating in τ and making ε tend to 0 we get finally (II.15) for a chosen such that $|Q(Y)| \leq a|Y|^2$. Lemma II.1 is proved. ■

From the monotonicity formula it is clear that, for any x_0 in M^4,

$$\lim_{r\to 0} \frac{1}{r^2} \int_{B^4_r(x_0)} |du|^2 \qquad \text{exists}$$

and we have

Theorem II.3 *Let u be a $J-$holomorphic map from (M^4, J) into $\mathbb{C}P^n$. Assume J is tamed by a symplectic form Ω and denote g the metric compatible with (Ω, J) (satisfying (I.4)), then there exists a closed subset $Sing\, u$ of M^4 such that*

i) $\mathcal{H}^2(Sing\, u) = 0$

ii) $u \in C^\infty(B^4 \setminus Sing\, u)$

iii) $Sing\, u = \{x_0 \in M^4 \quad ; \quad \lim_{r\to 0} \frac{1}{r^2} \int_{B^4_r(x_0)} |du|^2_g \geq \varepsilon_0\}$

where ε_0 is a positive constant depending only on (M^4, g).

Proof of theorem II.3. Since u minimizes the energy among approximable maps and since, in the comparison argument of R.Schoen and K.Uhlenbeck [SU] for proving the above $\varepsilon-$regularity result for minimizing harmonic map, the comparison map $u^{(h)}$ is smooth, the proof of theorem 3.1 of [SU] can be transposed here almost word by word to get theorem II.3 above. We could have also combine the stationarity property of $J-$holomorphic maps established in proposition II.2 with the more sophisticated $\varepsilon-$regularity result for stationary harmonic map of C.Evans in order to deduce the theorem II.3 straight. ■

The aim of this paper is to prove that whenever u is a $J-$holomorphic locally approximable map, Sing$u$, defined by iii), is made of isolated points. We first prove the following proposition

Proposition II.4 *Let u be a $J-$holomorphic locally approximable map from (M^4, J) into $\mathbb{C}P^n$. Let x_0 ∈Sing u, then, from any sequence $r_i \rightarrow 0$, one may extract a subsequence $r_{i'}$ such that, as $i' \rightarrow +\infty$,*

$$u_{r_{i'},x_0}(x) = u(r_{i'}x + x_0) \rightharpoonup u_\infty(x) = \phi \circ h_{x_0}(\frac{x}{|x|}) \text{ weakly in } W^{1,2}(B_1^4) \quad \text{(II.18)}$$

where we use normal coordinates about x_0 to define $r_i x + x_0$, ϕ is an holomorphic map from $\mathbb{C}P^1$ into $\mathbb{C}P^n$ and h_{x_0} is the Hopf fibration whose fiber are contained in J_{x_0} holomorphic 2-plane of $(\mathbb{R}^4, J_{x_0})$. Moreover the following measures of B_1^4 converge as Radon measures

$$|du_{r_{i'}}|^2 \, dx \rightharpoonup |du_\infty|^2 \, dx + \sum_{k=1}^{q} 8\pi \; N_k \; \mathcal{H}^2 \llcorner D_k \quad \text{(II.19)}$$

where dx denote the Lebesgue measure on $\mathbb{R}^4$, N_k are positive integers, D_k are $J_{x_0}-$holomorphic disks obtained by intersecting B_1^4 and $J_{x_0}-$holomorphic planes of $(\mathbb{R}^4, J_{x_0})$ and finally $\mathcal{H}^2 \llcorner D_k$ denotes the $2-$dimensional Hausdorff measure restricted to these disks.

See a Proof of the existence of tangent maps for stationary harmonic maps in [LT]. Note that the limiting configuration $(\phi, \{N_k, D_k\}_{k=1\ldots q})$ may a-priori depend on the extracted subsequence $r_{i'}$. As we will see in section III this is in fact not the case. Let ϕ be an holomorphic map from $\mathbb{C}P^1$ into $\mathbb{C}P^n$, this defines an element $[\phi]$ of $\pi_2(\mathbb{C}P^n) \simeq H_2(\mathbb{C}P^n; \mathbb{Z}) = \mathbb{Z}$. The Kähler form $\omega_{\mathbb{C}P^n}/4\pi$ integrated on $\mathbb{C}P^1 \subset \mathbb{C}P^n$ gives 1 and therefore generates $H^2(\mathbb{C}P^n;\mathbb{Z})$ and we have

$$\int_{\mathbb{C}P^1} \frac{|\nabla\phi|^2}{2} = \int_{\mathbb{C}P^1} \phi^*\omega_{\mathbb{C}P^n} = \omega([\phi]) \in 4\pi\mathbb{Z}$$

We call $deg\,\phi$ the integer $\omega([\phi])/4\pi$. From proposition (II.4) we will deduce the following corollary :

Corollary II.1 *Let u be a $J-$holomorphic map from (M^4,J) into $\mathbb{C}P^n$ and $x_0 \in Sing\,u$ then*

$$\lim_{r\rightarrow 0}\frac{1}{8\pi^2 r^2}\int_{B^4_r(x_0)}|du|^2 = N_{x_0} \in \mathbb{Z}^*_+ \tag{II.20}$$

N_{x_0} is called the multiplicity of u at x_0. If $(\phi,\{N_k,D_k\})$ is a limiting configuration given by proposition II.4 for some sequence $r_i \rightarrow 0$, then

$$N_{x_0} = deg\,\phi + \sum_{k=1}^{q} N_k \tag{II.21}$$

If we denote, for $N \in \mathbb{Z}^*_+$,

$$Sing^N u = \{x_0 \in Sing\,u \qquad s.\ t. \quad N_{x_0} \leq N\}$$

we have

$$Sing\,u \setminus Sing^N u \qquad \textit{is closed in } M^4$$

Proof of proposition II.4 : Let $x_0 \in \mathrm{Sing}\,u$ and $u_{r_i,x_0}(x) = u(r_i x + x_0)$ where we are using a given choice of normal coordinates (we may also have considered $u(\exp_{x_0}(r_i x))...$). We may extract a subsequence, still denoted u_{r_i,x_0}, such that

$$u_{r_i,x_0} \rightharpoonup u_\infty \qquad \text{weakly in } W^{1,2}$$

and

$$|du_{r_i,x_0}|^2\,dx \rightharpoonup |du_\infty|^2\,dx + \nu \qquad \text{as Radon measures}$$

where $\nu \perp dx$. From [Li] and [LR] there exists a 2-dimensional rectifiable subset Σ of M^4 and an integer valued function θ on Σ such that the defect measure ν writes

$$\nu = \theta\ \mathcal{H}^2 \llcorner \Sigma \tag{II.22}$$

(The fact that θ takes values in $8\pi\mathbb{Z}_+$, established in [LR], can be more directly established, arguing like in the proof of theorem 4.2.3 of [Ti], replacing somehow the identity $\int |F_A|^2 = \int \Omega\wedge Tr(F\wedge F)$ by the identity (II.8) satisfied by u : $\int |du|^2 = 2\int \Omega\wedge u^*\omega_{\mathbb{C}P^1}$). A simple scaling facts yields

$$\frac{1}{r_i^2}\int_{B_{r_i}(x_0)} |du|^2\ dvol_g = \int_{\tilde{B}_1(0)} |du_{r_i,x_0}|^2_{g_{r_i}}\ dvol_{g_{r_i}}$$

where $g_{r_i} = \Lambda^*_{r_i,x_0} g$ denotes the pull-back metric of g by the dilation $\Lambda_{r_i,x_0}(x) = r_i x + x_0$ and $\tilde{B}_1(0)$ the geodesic ball centered at 0 of radius 1 for g_{r_i}. The effect of this dilation on the monotonicity formula (II.15) implies that

$$\frac{\partial u_\infty}{\partial r} = 0 \tag{II.23}$$

Denote J_{r_i,x_0} the almost complex structure on $\tilde{B}^4_1$ obtained by dilating J by Λ_{r_i,x_0} (i.e. for any $Y \in T_x\tilde{B}^4_1$, $J_{r_i}Y = d\Lambda^{-1}Jd\Lambda.Y$), since u_{r_i,x_0} is $J_{r_i,x_0}-$holomorphic, and since the $J_{r_i,x_0}-$holomorphic equation passes easily to the limit under weak convergence in $W^{1,2}$ we get that u_∞ is $J_{x_0}-$holomorphic and solves

$$\forall X\in\mathbb{R}^4 \qquad du_\infty J_{x_0} X = j_0 du_\infty X \tag{II.24}$$

and in particular we deduce that

$$du_\infty J_{x_0}\frac{\partial}{\partial r} = 0 \tag{II.25}$$

We may always assume that, modulo a change of coordinates, J_{x_0} coincide with the standard complex structure on $\mathbb{R}^4 \simeq \mathbb{C}\oplus\mathbb{C}$ and then integral curves of $J_{x_0}\frac{\partial}{\partial r}$ are fibers of of the Hopf fibration

$$h\ :\ (z_1,z_2)\longrightarrow [z_1,z_2]$$

Thus combining (II.23) and (II.25) we can conclude that u_∞ is constant on the $J_{x_0}-$holomorphic disks obtained by intersecting the $J_{x_0}-$holomorphic plane with the unit ball B^4_1. Therefore u_∞ writes $u_\infty(x) = \phi\circ h\left(\frac{x}{|x|}\right)$ where, because of (II.24), ϕ is holomorphic from $\mathbb{C}P^1$ into $\mathbb{C}P^n$. This implies in particular that, using the notation $\hat{h}(x) = h(\frac{x}{|x|})$,

$$\int_{B_1^4} |\nabla u_\infty|^2 = \int_{\mathbb{C}P^1} dp \int_{\hat{h}^{-1}(p)} |\nabla\phi|^2(p)\ \frac{|\nabla\hat{h}|^2}{2|J_2\hat{h}|}$$
$$= \int_{\mathbb{C}P^1} \pi|\nabla\phi|^2(p)\ dp = 8\pi^2 \deg\phi$$

We prove now that Σ is a union of $J_{x_0}-$holomorphic disks obtained by intersecting B_1^4 with $J_{x_0}-$holomorphic planes of $\mathbb{R}^4$.

First of all, following [Li], for $\mathcal{H}^2-$a.e. $y \in \Sigma$ there exists y_i converging to y and σ_i tending to 0 such that $u_{r_i}(\sigma_i x + y_i)$ converges strongly to $u_y(x)$ (a "bubble" at y) which is obtained by translating a non-constant harmonic map from $\mathbb{C}$ into $\mathbb{C}P^n$ along the directions parallel to $T_y\Sigma$. Since u_y is $J_{x_0}-$holomorphic we deduce that $T_y\Sigma$ is a $J_{x_0}-$holomorphic plane. Therefore Σ is a $J_{x_0}-$holomorphic rectifiable subset of B^4.

Arguing like in [LR] (proof of theorem B), using the stationary equation (II.16) we can prove that the measure

$$\mu = \lim_{i\rightarrow+\infty} |\nabla u_{r_i,x_0}|^2\ dvol_{g_{r_i}}$$

is invariant under radial deformation : for any measurable subset A and $1 > \lambda > 0$

$$\mu(\lambda A) = \lambda\mu(A) \tag{II.26}$$

Since this is the case for $|\nabla u_\infty|^2\ dx$ too we deduce that Σ is a cone centered at x_0. It remains to prove that a 2 dimensional locally finite mass $J_{x_0}-$holomorphic rectifiable cone centered at the origin is made of a union of $J_{x_0}-$holomorphic planes passing through the origin. Since J_{x_0} is integrable we can use complex coordinates to verify this elementary fact.

Finally we claim that θ is constant on each of the concentration disk D_k and equal to 8π times an integer. Let $p \in \Sigma\setminus\{x_0\}$, denote $D^2_{y,\alpha}$ the disk perpendicular to $T_y\Sigma$ of radius α. α is chosen small enough so that $D^2_{y,\alpha}\cap\Sigma = \{y\}$. Let $z \in \Sigma\setminus\{x_0\}\cap S_y$ where S_y is the circle deduced by integrating the vector field $J_{x_0}\frac{\partial}{\partial r}$ starting from y. Denote $D^2_{z,\alpha}$ the disk passing through z perpendicular to Σ of radius α. It is obtained, in particular, by integrating the trajectories issued from $D^2_{y,\alpha}$ along the vector-field $J_{x_0}\frac{\partial}{\partial r}$ until reaching the 2-plane perpendicular to $T_z\Sigma$. Denote by $N^3_{y,z}$ the 3-dimensional submanifold of B^4 made of this trajectories. Since $d(u^*_{r_i,x_0}\omega_{\mathbb{C}P^n})=0$, we have

$$0=\int_{N^3_{y,z}} d(u^*_{r_i,x_0}\omega_{\mathbb{C}P^n})=\int_{D^2_{y,\alpha}} u^*_{r_i,x_0}\omega_{\mathbb{C}P^n}$$
$$-\int_{D^2_{z,\alpha}} u^*_{r_i,x_0}\omega_{\mathbb{C}P^n}+\int_{\partial N^3\setminus D^2_{y,\alpha}\cup D^2_{z,\alpha}} u^*_{r_i,x_0}\omega_{\mathbb{C}P^n}$$

Because of the analysis in [LR] we have

$$\int_{D^2_{y,\alpha}} u^*_{r_i,x_0}\omega_{\mathbb{C}P^n}\longrightarrow 4\pi\sum \text{ deg. of the bubbles at } y$$

$$\int_{D^2_{z,\alpha}} u^*_{r_i,x_0}\omega_{\mathbb{C}P^n}\longrightarrow 4\pi\sum \text{ deg. of the bubbles at } z$$

and

$$\int_{\partial N^3\setminus D^2_{y,\alpha}\cup D^2_{z,\alpha}} u^*_{r_i,x_0}\omega_{\mathbb{C}P^n}\longrightarrow\int_{\partial N^3\setminus D^2_{y,\alpha}\cup D^2_{z,\alpha}} u^*_{\infty}\omega_{\mathbb{C}P^n}$$

where we are using that $\partial N^3\setminus D^2_{\alpha,y}\cup D^2_{\alpha,z}$ is disjoint from Σ and we have a C^k convergence of u_{r_i,x_0} on this set. Since $\partial N^3\setminus D^2_{y,\alpha}\cup D^2_{z,\alpha}$ contains $J_{x_0}\frac{\partial}{\partial r}$ in its tangent plane, we have

$$\int_{\partial N^3\setminus D^2_{y,\alpha}\cup D^2_{z,\alpha}} u^*_{\infty}\omega_{\mathbb{C}P^n}=0$$

and we deduce that $4\pi\sum($ deg. of the bubbles at $y)=4\pi\sum($ deg. of the bubbles at $z)$. Using the fact that

$$4\pi\sum(\text{ deg. of the bubbles at } y)=\frac{1}{2}\sum(\text{energies of bubbles at } y)=\frac{\theta(y)}{2}$$

we have proved that $\theta(y)=\theta(z)\in 8\pi\mathbb{Z}_+$ for any y and z lying on the same circle integrating the vector-field $J_{x_0}\frac{\partial}{\partial r}$. Because of (II.26) θ is invariant along the $\frac{\partial}{\partial r}$ direction too. Therefore we have proved proposition II.4. ∎

Let u be a $J-$holomorphic map from (M^4,J) into $\mathbb{C}P^1$. Let p be a regular value of u in $M^4\setminus\mathrm{Sing}\,u$ such that $u^{-1}(p)$ is a smooth submanifold of $M^4\setminus\mathrm{Sing}\,u$ verifying $\mathcal{H}^2(u^{-1}(p))<+\infty$. Since $u\in W^{1,2}(M^4,\mathbb{C}P^1)$ and since

u is smooth apart from a $\mathcal{H}^2$ measure 0 set, this holds for almost every p in $\mathbb{C}P^1$. For such a regular p we denote by u_p the following rectifiable current

$$u_p = \left\langle \left[B^4 \setminus \mathrm{Sing}\, u\right] ; u ; p\right\rangle \tag{II.27}$$

where we adopt the notations of [Fe] : for N an oriented submanifold, $[N]$ denotes the integration on N and for g a given map on M and q a regular value for g, $\langle [N], g, q\rangle$ is the slice of $[N]$ in $g^{-1}(q)$ - See [Fe] 4.2.1. Granting these notations u_p corresponds therefore to the rectifiable current carried by $u^{-1}(p) \cap M^4 \setminus \mathrm{Sing}\, u$ oriented by $\frac{u^*\omega_{\mathbb{C}P1}}{|u^*\omega_{\mathbb{C}P1}|}$ of multiplicity 1. We have the following proposition.

Proposition II.5 *Let u be a locally approximable $J-$holomorphic map from (M^4, J) into $\mathbb{C}P^1$. Then for almost every $p \in \mathbb{C}P^1$, the rectifiable current u_p defined by (II.27) is a $J-$holomorphic integral rectifiable cycle :*

i) $\partial u_p = 0$

ii) for $\mathcal{H}^2-$a.e. x in $u^{-1}(p) \cap M^4 \setminus Sing\, u$, $T_x u^{-1}(p)$ is a J_x-holomorphic 2-plane.

Proof of proposition II.5 : Let $\eta \in C^\infty(S^2, \mathbb{R}_+)$ whose support is contained in the north hemisphere S^2_+. We also choose η to be axially symmetric with respect to rotations about the vertical axis Oz and "normalized" so that $\int_{S^2} \eta = 1/4\pi$. For any $p \in S^2$ and $\varepsilon > 0$ small, we denote $\eta_p^\varepsilon \in C^\infty(S^2, \mathbb{R}_+)$ the map given by

$$\eta_p^\varepsilon(x) = \frac{1}{\varepsilon^2}\eta^\varepsilon(R_{\theta(p)}x) \tag{II.28}$$

where $R_{\theta(p)}$ is any rotation of $SO(3)$ sending the north pole to p, and $\eta^\varepsilon = \eta(x/\varepsilon)$, where x/ε denotes the point on the geodesic passing through x, coming from the north pole, distant from the north pole of a distance =dist(x,North)$/\varepsilon$. We have of course $\int_{S^2} \eta_p^\varepsilon = 1/4\pi$ and $\mathrm{Supp}\, \eta_p^\varepsilon \subset B^2_{\frac{\pi\varepsilon}{2}}(p)$ the geodesic ball of center p and radius $\frac{\pi\varepsilon}{2}$ in S^2.

We claim that for any $\psi \in C_0^\infty(\wedge^2 M^4)$ and for almost every p in S^2 we have

$$\langle u_p;\psi\rangle := \int_{u^{-1}(p)} \psi \cdot \frac{u^*\omega_{\mathbb{C}P^1}}{|u^*\omega_{\mathbb{C}P^1}|}\, d\mathcal{H}^2 \llcorner u^{-1}(p) = \lim_{\varepsilon\to 0}\int_{B^4} \psi \wedge u^*(\eta_p^\varepsilon \omega_{\mathbb{C}P^1}) \tag{II.29}$$

Let K be a compact regular subset of the open set $B^4 \setminus \mathrm{Sing}\, u$. By the mean of the coarea formula one easily proves that, for any regular point of u,

$$\langle u_p \llcorner K; \psi\rangle = \lim_{\varepsilon\to 0} \int_K \psi \wedge u^*(\eta_p^\varepsilon \omega_{\mathbb{C}P^1})$$

Let $\{S_k\}_{k=1\ldots\infty}$ be a family of open sets such that $S_k \subset S_{k-1}$ and $\cap_{k=1}^\infty S_k = \mathrm{Sing}\, u$, so that

$$\lim_{N\to+\infty} |\cap_{k\le N} S_k| = 0$$

Since $\int_{B^4} |du|^2 < +\infty$, using the coarea formula, we have that $\mathcal{H}^2(u^{-1}(p)) \in L^1(\mathbb{C}P^1)$ so that $\mathcal{H}^2(u^{-1}(p)) < +\infty$ for a.e. $p \in \mathbb{C}P^1$ and we shall consider these p only. For p which are Lebesgue points of the function $p \to \mathcal{H}^2(u^{-1}(p))$ we have

$$\begin{aligned} \mathcal{H}^2(u^{-1}(p)\cap S_k) &= \lim_{\varepsilon\to 0} \frac{1}{|B_\varepsilon^2(p)|} \int_{B_\varepsilon^2(p)} \mathcal{H}^2(u^{-1}(q)\cap S_k)\ dq \\ &= \lim_{\varepsilon\to 0} \int_{\mathbb{C}P^1} \eta_p^\varepsilon(q)\ \mathcal{H}^2(u^{-1}(q)\cap S_k)\ dq \end{aligned} \tag{II.30}$$

Moreover, since $\mathcal{H}^2(\overline{\mathrm{Sing}\, u}) = 0$, $\cap S_k = \mathrm{Sing}\, u$ we have

$$\lim_{k\to+\infty} \mathcal{H}^2(u^{-1}(p)\cap S_k) = 0 \tag{II.31}$$

combining (II.30) and (II.31) we get that, for such a p, there exists $\varepsilon_p(k) > 0$ such that

$$\begin{aligned} &\limsup_{k\to+\infty} \sup_{\varepsilon<\varepsilon_p(k)} \int_{S_k} \eta_p^\varepsilon(u)\ |u^*\omega_{\mathbb{C}P^1}| \\ &\quad \le \limsup_{k\to+\infty} \sup_{\varepsilon<\varepsilon_p(k)} \int_{\mathbb{C}P^1} \eta_p^\varepsilon(q)\ \mathcal{H}^2(u^{-1}(q)\cap S_k) = 0 \end{aligned}$$

Let $\delta > 0$ and choose k (whose existence is given by (II.31)) such that

$$|\langle u_p \llcorner S_k; \psi\rangle| \le \delta$$

and

$$\sup_{\varepsilon<\varepsilon_p(k)} \eta_p^\varepsilon(u)\ |u^*\omega_{\mathbb{C}P^1}| \le \delta$$

For every $\varepsilon > 0$, we have

$$\begin{aligned}
&\left|\int_{B^4} u^*(\eta_p^\varepsilon \omega_{\mathbb{C}P^1}) \wedge \psi - \int_{u^{-1}(p)\cap B^4\setminus \mathrm{Sing}\, u} \psi\right| \\
&\leq C\int_{S_k} \eta_p^\varepsilon(u)\ |u^*\omega_{\mathbb{C}P^1}| + |\,\langle u_p \llcorner S_k ; \psi\rangle\,| \\
&+\left|\int_{B^4\setminus S_k} u^*(\eta_p^\varepsilon \omega_{\mathbb{C}P^1}) \wedge \psi - \int_{u^{-1}(p)\cap B^4\setminus S_k} \psi\right| \leq 3\delta
\end{aligned}$$

for ε small enough. The claim (II.29) is then proved.

Since $d(u^*\eta_p^\varepsilon \omega_{\mathbb{C}P^1}) = 0$, we deduce from (II.29) that u_p is a rectifiable cycle.

Take now any point x in $u^{-1}(p) \cap B^4 \setminus \mathrm{Sing}\, u$, we have

$$T_x u^{-1}(p) = \mathrm{Ker}\, u^*\omega_{\mathbb{C}P^1} = \mathrm{Ker}\, du$$

But, from the equation (I.1), we deduce that $\mathrm{Ker}\, du$ is invariant by J, therefore $T_x u^{-1}(p)$ is a J_x holomorphic 2-plane and Proposition II.5 is proved. ∎

We recall some basic facts on intersection of surfaces in B^4 (see [Fe] 4.1.9 and 4.3.20). Let Σ_1 and Σ_2 be two oriented surfaces and f_1 and f_2 immersions of these two surfaces in B^4 such that $f_1(\partial\Sigma_1) \cap f_2(\partial\Sigma_2) = \emptyset$. f_1 and f_2 are homotopic (preserving $f_i(\partial\Sigma_i)$ fixed) to transverse immersions g_1 and g_2. At $(p_1, p_2) \in \Sigma_1 \times \Sigma_2$ such that $g_1(p_1) = g_2(p_2)$, we denote by $\delta(p_1, p_2)$ the sign of intersection of $g_1(\Sigma_1)$ and $g_2(\Sigma_2)$ at (p_1, p_2).

Denote $S(g_1, g_2) = \{(p_1, p_2) \quad \text{s. t.} \quad g_1(p_1) = g_2(p_2)\}$ it is a standard fact that

$$\sum_{(p_1,p_2)\in S(g_1,g_2)} \delta(p_1, p_2)$$

is independent on the choice of (g_1, g_2) transverse to each other and homotopic to (f_1, f_2). We denote it by $f_1 \cdot f_2$ (or also by abuse of notation $f_1(\Sigma_1)\ \cdot\ f_2(\Sigma_2)$ whenever the immersions are implicit).

Assume (p_1, p_2) is an isolated point of $S(f_1, f_2)$, we may choose U, neighborhood of $f_1(p_1)$ in B^4, such that, if W_i denotes the connected component of $f_i^{-1}(U)$ in Σ_i containing p_i, then $f_1|W_1\ \cdot\ f_2|W_2$ is independent of U and we denote it $\delta_{f_1 f_2}(p_1, p_2)$ (or simply $\delta(f_1, f_2)$ when there is no ambiguity).

For any regular value p of u, $u^{-1}(p)\cap B^4\setminus\mathrm{Sing}\,u$ is a smooth $J-$holomorphic embedded surface. It inherits a complex structure from J and we therefore interpret $u^{-1}(p)\cap B^4\setminus\mathrm{Sing}\,u$ as an embedded $J-$holomorphic complex surface $f_p\ :\ \Sigma_p\longrightarrow u^{-1}(p)\cap B^4\setminus\mathrm{Sing}\,u$ via the canonical inclusion map. The following proposition holds.

Proposition II.6 *Let p be any regular value of u in $\mathbb{C}P^1$. Denote $f_p\ :\ \Sigma_p\rightarrow u^{-1}(p)\cap B^4\setminus Sing\,u$ the tautological $J-$holomorphic embedding realizing $u^{-1}(p)\cap B^4\setminus Sing\,u$. Let $f\ :\ D^2\rightarrow B^4$ be an immersed $J-$holomorphic disk such that*

i) $\overline{D}:=\overline{f(D^2)}\subset B^4\setminus Sing\,u$

ii) $\partial D\cap\overline{u^{-1}(p)}=\emptyset$

then $f_p^{-1}(u^{-1}(p)\cap B^4\setminus Sing\,u\cap D)$ and $f^{-1}(u^{-1}(p)\cap B^4\setminus Sing\,u\cap f(D))$ are isolated points of Σ_p and D^2. Moreover, if $f_p(x)=f(y)$, $\delta_{f_pf}(x,y)\geq 1$ and we have

$$<[B^4\setminus Sing\,u],u,p>\ \cdot[D]:=\sum_{(x,y)\ s.t.\ f_p(x)=f(y)}\delta_{f_pf}(x,y)=\lim_{\varepsilon\rightarrow 0}\int_{f(D)}\eta_p^\varepsilon(u(x))\ u^*\omega \tag{II.32}$$

where η_p^ε is given by (II.28).

Remark II.2 *Observe also that from standard intersection theory ([Fe] 4.3.20), if $f'\ :\ D^2\rightarrow B^4$ is another immersed disk, $D':=f'(D^2)$, which is cobordant to $D=f(D^2)$ in $B^4\setminus\overline{u^{-1}(p)}$ (i.e. there exists a smooth surface Σ in $B^4\setminus\overline{u^{-1}(p)}$ such that $\partial\Sigma=\partial D-\partial D'$), then*

$$<[B^4\setminus Sing\,u],u,p>\ \cdot[D']\llcorner B^4=<[B^4\setminus Sing\,u],u,p>\ \cdot[D]\llcorner B^4 \tag{II.33}$$

Proof of proposition II.6 : The first statement of this proposition is a direct consequence of Micallef and White's result on intersection of smooth $J-$holomorphic surfaces (see theorem 7.1 of [MW]). (II.32) is straightforward whenever D and $u^{-1}(p)$ are transverse to each other. When it is not the case, we take $g\ :\ D^2\rightarrow B^4$ (denoting $D_g:=g(D^2)$) such that $\partial D_g:=g(\partial D^2)=\partial D$, homotopic to f_p by an homotopy leaving the boundary fixed and such that g is transverse to $u^{-1}(p)$. Clearly we have, using successively (II.33) and the coarea formula

$$< [B^4\backslash \mathrm{Sing}\, u], u, p > \ \cdot [D] =< [B^4\backslash \mathrm{Sing}\, u], u, p > \ \cdot D_g = \lim_{\varepsilon\rightarrow 0}\int_{D_g} \eta_p^\varepsilon(u(x))\, u^*\omega$$

Moreover, let Δ^3 a 3-immersed manifold such that $\partial\Delta^3 = D - D_g$, since $d[\eta_p^\varepsilon(u(x)]\ u^*\omega) = 0$, we have

$$\begin{aligned}\int_D \eta_p^\varepsilon(u(x))\ u^*\omega \ &= \int_{D_g} \eta_p^\varepsilon(u(x))\ u^*\omega + \int_{\Delta^3} d(\eta_p^\varepsilon(u(x))\ u^*\omega)\\ &= \int_{D_g} \eta_p^\varepsilon(u(x))\ u^*\omega\end{aligned}$$

which proves (II.32) in any cases. ■

Proposition II.7 *(Almost uniqueness of the tangent cone) For any $x_0 \in M^4$, using normal coordinates about x_0, denote $u_{r,x_0}(x) = u(rx + x_0)$ and assume there exists $r_i \rightarrow 0$ such that u_{r_i,x_0} converges weakly to some map u_∞ which is not constant, then the whole sequence $u_{r,x_0}(x) = u(rx + x_0)$ converges to u_∞ moreover it is a $J_{x_0}-$holomorphic map of the form $u_\infty(x) = \phi\circ h\left(\frac{x}{|x|}\right)$ where h is the hopf fibration from ∂B^4 into $\mathbb{C}P^1$ whose fibers are contained in $J_{x_0}-$holomorphic 2-planes and ϕ is an holomorphic map from $\mathbb{C}P^1$ into $\mathbb{C}P^1$ of positive degree. Moreover, in any cases, there exist $D_1\cdots D_q$, $J_{x_0}-$holomorphic disks such that for any $r_i \rightarrow 0$ there exists a subsequence $r_{i'}$ and $N_1\cdots N_q$ q positive integers such that*

$$|du_{r_{i'},x_0}|^2\ dx \rightharpoonup |du_\infty|^2 dx + 8\pi\sum_{k=1}^q N_k\ \mathcal{H}^2 \llcorner D_k$$

as Radon measures ($|du_\infty|^2$ has to be replaced by 0 in the case where du converges weakly to 0).

Remark II.3 *The important fact here is that $N_1\cdots N_q$ may a-priori depend on r_i and the extracted subsequence $r_{i'}$ but q and $D_1\cdots D_q$ are independent on the chosen sequence r_i. As we will see in part 3, $N_1\cdots N_q$ won't depend on the sequence r_i neither but this requires more work.*

Proof of proposition II.7 : Of course a proof is needed only if $x_0 \in \operatorname{Sing} u$. Let $N_{x_0} := \lim_{r\to 0} \frac{1}{r^2} \int_{\tilde{B}^4_r} |du|^2$ the non-zero integer given by the corollary II.1. From proposition II.4 we know that, from a given sequence $r_i \to 0$, we may always extract a subsequence $r_{i'}$ such that

$$u_{r_{i'},x_0} \rightharpoonup u_\infty \qquad \text{weakly in } W^{1,2}$$

and

$$|du_{r_{i'},x_0}|^2\ dx \rightharpoonup |du_\infty|^2\ dx + 8\pi \sum_{k=1}^{q} N_k\ \mathcal{H}^2 \llcorner D_k$$

Assume we may find two different sequences $r_i^1 \to 0$ and $r_i^2 \to 0$ such that, the limiting configurations are different :

$$(u^1_\infty, \{D^1_k\}_{k=1\cdots q^1}) \neq (u^2_\infty, \{D^2_k\}_{k=1\cdots q^2})$$

From proposition II.4, the ∂D^j_k, for $j = 1, 2$, are fibers of the Hopf fibration h. Let U be the open set in $\mathbb{C}P^1$ obtained by removing the finite set of points given by $\phi^j(h(\partial D^j_k))$ for $j = 1, 2$, $k = 1 \cdots N^1$ and $k = 1 \cdots N^2$. Let p in U being a regular value simultaneously of all the $u_{r^1_i,x_0}$'s (countably many) and all the $u_{r^2_i,x_0}$'s (countably many also) including the limits u^j_∞. Because of the strong convergence in $W^{1,2}$ of the $u_{r^j_i,x_0}$ to u^j_∞ in $B^4 \setminus \cup_{k=1}^{q^j} D^j_k$, since $u_{r^j_i,x_0}$ are harmonic maps, since also $u^j_\infty \in C^\infty(B^4 \setminus \{0\})$, the convergence of the $u_{r^j_i,x_0}$ to u^j_∞ in $B^4 \setminus \cup_{k=1}^{q^j} D^j_k$ holds in the $C^l_{loc}-$topology on $B^4_1 \setminus \cup_{k=1}^{q^j} D^j_k$ for any integer $l \geq 0$. Therefore, for any $\varepsilon > 0$, there exists $i_0 \in \mathbb{N}$ such that, for $i \geq i_0$,

$$u^{-1}_{r^j_i}(p) \subset T^j_\varepsilon \tag{II.34}$$

where

$$T^j_\varepsilon = \left\{ x \quad \text{s.t.} \quad dist\left(x; (u^j_\infty)^{-1}(p) \cup \cup_{k=1}^{q^j} D^j_k\right) \leq \varepsilon \right\} \quad .$$

Modulo a change of coordinates we can identify J_{x_0} with the standard complex structure on $\mathbb{R}^4 \simeq \mathbb{C}\oplus\mathbb{C}$ and consider a disk $D = \{(w, z) \quad \text{s.t. } aw + bz = 0\}$ different from any of the D^j_k and the components of $(u^j_\infty)^{-1}(p)$. We claim that for i large enough

$$< B^4, u_{r^j_i}, p > \ \cdot [D] = N_{x_0} \tag{II.35}$$

Keeping the boundary of D fixed, we may homotope D into $\tilde{D}$ such that $\tilde{D}\subset B^4\setminus\{0\}$ $\tilde{D}$ is transverse to the D^j_k, transverse to $(u^j_\infty)^{-1}(p)$ (which is made of finitely many holomorphic flat disks in B^4). For Hopf degree reasons

$$< \overline{B}^4, u^j_\infty, p > \ \cdot [\tilde{D}] = \text{Hopf degree of } u^j_\infty|\partial B^4 = \deg\phi \tag{II.36}$$

Denote

$$S^j_\varepsilon = \left\{x \quad \text{s.t.} \quad \text{dist}\,\left(x, (u^\infty_j)^{-1}(p)\right) \leq \varepsilon\right\}$$

using now (II.34), for ε small enough and i large enough, because of (II.36) and the strong convergence of $u^j_{r_i}$ to u^j_∞ in the neighborhood of $\tilde{D}\cap(u^j_\infty)^{-1}(p)$ that avoids 0 and the D^j_k (because p is in U), we have

$$< B^4, u_{r^j_i}, p > \llcorner S^j_\varepsilon \cdot [\tilde{D}]\llcorner S^j_\varepsilon = \deg\phi \tag{II.37}$$

Denote

$$R^j_\varepsilon = \left\{x \quad \text{s.t.} \quad \text{dist}\,\left(x, \cup_{k=1}^{q^j} D^j_k\right) \leq \varepsilon\right\}$$

For ε small enough $\tilde{D}$ intersects the R^j_ε into q^j disjoint disks Δ^j_k. From the proof of proposition II.4 and corollary II.1, we know that N^j_k corresponds to the limiting topological degree of $u_{r^j_i,x_0}$ restricted to Δ^j_k into $\mathbb{C}P^1$, therefore, for i large enough we have

$$< B^4, u_{r^j_i,x_0}, p > \llcorner \Delta^j_k \cdot [\tilde{D}]\llcorner \Delta^j_k = N^j_k \tag{II.38}$$

Using (II.21), combined with (II.37) and (II.38) we get (II.35).

Assume first that $u^1_\infty \neq u^2_\infty$, then we may find a regular p in U such that $(u^1_\infty)^{-1}(p)\neq(u^2_\infty)^{-1}(p)$: there exists an holomorphic flat disk $D^\star$ included in $(u^1_\infty)^{-1}(p)\cup\{0\}$, passing through the origin, whose intersection with $(u^2_\infty)^{-1}(p)$ is empty (0 does not belong to $(u^2_\infty)^{-1}(p)$). Let $r^2_{i_0} > r^1_{i_1}$, both i_0 and i_1 to be chosen large enough later. Pick any $y=(w_y,z_y)$ in $D^\star\cap B^4_{r^1_{i_1}}\setminus B^4_{r^1_{i_1}/2}$. There exists at least one point $y'=(w_{y'},z_{y'})$ in $u^{-1}(p)$ contained in the disk perpendicular to $D^\star$ at y and whose distance to $D^\star$ is $|y-y'| = o(r^1_{i_1})$ and the holomorphic disk $D^{\star\prime}=\{(w,z) \quad \text{s.t.} \quad z_{y'}w-w_{y'}z = 0\}$ converges to $D^\star$. By the mean of lemma A.3 we perturb $D^{\star\prime}$ into a $J_{r^2_{i_0},x_0}-$holomorphic disk $\tilde{D}^\star$ passing through 0 and the point $y'/r^2_{i_0}$. By taking $r^2_{i_0}$ and $r^1_{i_1}$ smaller and smaller, since $J_{r^2_{i_0},x_0}$ converges in any C^l-norm to the integrable standard complex structure on B^4, $\tilde{D}^\star$ is as close as we want from $D^\star$ in C^l-norm. Since $T_0\tilde{D}^\star \simeq D^\star$ is disjoint from $(u^2_\infty)^{-1}(p)$ and the D^2_k for $k=1...q^2$ (recall that p has been chosen in U so that $(u^j_\infty)^{-1}(p)\cap D^{1,2}_k=\emptyset$), considering $r^2_{i_2}<r^1_{i_1}/2$ on B^4_1, we have from (II.35)

$$
\begin{aligned}
&< [B^4_{r^2_{i_2}}], u, p > \cdot[\tilde{D}^\star]\llcorner B^4_{r^2_{i_2}} =< [B^4_1], u_{r^2_{i_2}}, p > \cdot \left[\frac{1}{r^2_{i_2}}\tilde{D}^\star\right]\llcorner B^4_1 \\
&=< B^4_1, u_{r^2_{i_2}}, p > \cdot [D^\star]\llcorner B^4_1 = N_{x_0}
\end{aligned}
\tag{II.39}
$$

where $\frac{1}{r^2_{i_2}}\tilde{D}^\star$ is the immersion that gives $\tilde{D}^\star$ in $\mathbb{R}^4$ multiplied by the factor $\frac{1}{r^2_{i_2}}$. But since we have that

$$
y' \in \tilde{D}^\star \cap B^4_{r^1_{i_1}} \setminus B^4_{r^2_{i_2}} \neq \emptyset \tag{II.40}
$$

and since both $u^{-1}_{r^2_{i_0},x_0}(p)\setminus \mathrm{Sing}\, u$ and $\tilde{D}$ are $J_{r^2_{i_0},x_0}-$holomorphic, from Micallef and White's result [MW], they intersect each other in a positive way, therefore, combining (II.39) and (II.40) we get

$$
< [B^4_{r^2_{i_0}}], u, p > \cdot[\tilde{D}^\star]\llcorner B^4_{r^2_{i_0}} > N_{x_0} \tag{II.41}
$$

Having chosen r^2_{i0} small enough the same identity (II.39) as the one established for $r^2_{i_2}$ should hold for $r^2_{i_0}$ too, we have then a contradiction and $u^1_\infty = u^2_\infty$. The proof that $\{D^1_k\}_{k=1\cdots q^1} = \{D^2_k\}_{k=1\cdots q^2}$ is established using the same arguments. Proposition II.7 is then proved. ■

III Proof of theorem I.1 for maps to $N=\mathbb{C}P^1$

In order to prove theorem I.1 for maps to $N=\mathbb{C}P^1$ we will prove by induction the following proposition.

Proposition III.1 *Let u be a locally approximable $J-$holomorphic map from (B^4,J) into $\mathbb{C}P^1$ (J tamed by a symplectic form Ω on B^4), then, for any $N>0$, $Sing^N u$ is made of isolated points in $B^4\setminus(Sing\, u\setminus Sing^N u)$. Moreover, for almost every $p\in\mathbb{C}P^1$, $\overline{u^{-1}(p)}\setminus(Sing\, u\setminus Sing^N u)$ is parametrized by a smooth $J-$holomorphic curve of $B^4\setminus(Sing\, u\setminus Sing^N u)$.*

III.1 Proof of proposition III.1 for $N = 1$

Let $x_0 \in \mathrm{Sing}^1 u$, we shall prove that x_0 is an isolated point of $\mathrm{Sing}\, u$. Precisely we will prove that there exists a neighborhood $B_\rho(x_0)$ of x_0 such that, for almost every p, $u^{-1}(p) \cap B_\rho(x_0)$ is a smooth $J-$holomorphic curve.

Since any point in $\mathrm{Sing}\, u$ is in the closure of $u^{-1}(q)$ for any q in S^2 and since, from [MW], two disjoint $J-$holomorphic disks intersect each other along isolated points, the smoothness of $u^{-1}(p) \cap B_\rho(x_0)$ for almost every p will imply that x_0 is an isolated point of $\mathrm{Sing}\, u$.

From proposition II.7 we have that, either $u_{r,x_0} \to u_\infty$ strongly in $W^{1,2}$, where $u_\infty = \phi \circ h(\frac{x}{|x|})$ and ϕ is a degree 1 $J_{x_0}-$holomorphic map from $\mathbb{C}P^1$ into $\mathbb{C}P^1$ or du_{r,x_0} converges weakly to 0 and $|du_{r,x_0}|^2\, dx \to 8\pi\mathcal{H}^2 \llcorner D$, where D is a $J_{x_0}-$holomorphic disk. In the first case, because of the strong convergence in $W^{1,2}$ of u_{r,x_0} to u_∞ which is smooth apart from the origin, we would get, using the $\varepsilon-$regularity (theorem II.3 above) that u_{r,x_0} would converge in C^k norm of $B^4_1 \setminus B^4_\delta$ for any $\delta > 0$. From such a convergence we deduce straight that x_0 is an isolated singularity of u.

Therefore in order to prove proposition III.1 for $N = 1$, we restrict now to the case where $|du_{r,x_0}|^2\, dx \to 8\pi\mathcal{H}^2 \llcorner D$ where D is a $J_{x_0}-$holomorphic flat 2-disk. For the convenience of the presentation we change now the chart so that in the new coordinates J_{x_0} becomes the canonical complex structure J_0 on $\mathbb{R}^4 \simeq \mathbb{C} \oplus \mathbb{C}$ and also so that D becomes the "horizontal" disk given by $\{(w, z) \quad \text{s.t.} \quad z = 0\}$. By the mean of Lemma A.4 we can, for r small enough, foliate $D^2_1 \times D^2_1$ by J_r-holomorphic disks which are almost "vertical". Precisely, we have the existence of a diffeomorphism

$$\begin{array}{rcl} \Theta \ : \ V & \longrightarrow & D^2_1 \times D^2_1 \\ (w, z) & \longrightarrow & (\theta_w(z), z) \end{array}$$

such that $\theta_w(0) = w$, $\Theta_{w_0} = \{(\theta_{w_0}(z), z)\}$ is a J_r-holomorphic disk for any fixed w_0 and where V is an open set of $\mathbb{C} \oplus \mathbb{C}$ containing the origin. From proposition II.4 we can extract a sequence r_i such that $u_{r_i,x_0} \to$ constant in $C^l_{loc}(B^4_1 \setminus D)$ for any integer l and the constant may be assumed to be the north-pole of $\mathbb{C}P^1 \simeq S^2$ (to fix ideas). Therefore for any regular value p of u, different from the north-pole, for r small enough, $(u_{r_i,x_0})^{-1}(p) \setminus \mathrm{Sing}\, u$ has an intersection number with Θ_{w_0} disks equal to $+1$ for any w_0. Since $\mathcal{H}^2(\mathrm{Sing}\, u) = 0$, apart from a measure zero set θ_w has some intersection point with $(u_{r_i,x_0})^{-1}(p)\setminus \mathrm{Sing}\, u$. Since both Θ_{w_0} and $(u_{r_i,x_0})^{-1}(p)\setminus \mathrm{Sing}\, u$ are J_r-holomorphic they intersect each other along exactly one point for almost every w_0. Call this point $\Phi_{p,r_i}(w_0)$. Denote $\phi_{p,r_i}(w_0)$ the complex function given by $\Theta^{-1}\circ\Phi_{p,r_i}(w_0)=(w_0,\phi_{p,r_i}(w_0))$. We pull back the almost-complex structure J_{r_i,x_0} by Θ and we keep denoting it by J_{r_i,x_0}. Note that we keep having as r_i tends to 0

$$J_{r_i,x_0}\longrightarrow J_0 \qquad \text{in } C^l \quad \forall l\geq 0$$

We claim that, for r_i small enough, ϕ_{p,r_i} is a Lipschitz complex function on $D^2_{\frac{1}{2}}$. Indeed, assume we would find two generic points w_1 and w_2 in $D^2_{\frac{1}{2}}$ such that

$$|\phi_{p,r_i}(w_1)-\phi_{p,r_i}(w_2)|>2|w_1-w_2|$$

By the mean of Lemma A.3 we produce a $J_{r_i,x_0}-$holomorphic disk approximating $H^0(z)=(h^0(z),z)$ where

$$h^0(z)=w_2-(z-\phi_{p,r_i}(w_1))\ \frac{w_1-w_2}{\phi_{p,r_i}(w_1)-\phi_{p,r_i}(w_2)}$$

which is transverse to D with an angle bounded from below by a fixed number. Thus by a straightforward intersection theory we have

$$[H^0]\llcorner B^4_1\cdot\ < B^4_1,u_{r_i,x_0},p>=+1$$

But, by construction , H^0 intersects $(u_{r_i,x_0})^{-1}(p)\setminus\mathrm{Sing}\, u$ at least twice at $(w_1,\phi_{p,r_i}(w_1))$ and at $(w_2,\phi_{p,r_i}(w_2))$. This contradicts Micallef and White's result on the positivity of the intersections of $J-$holomorphic curves. Thus Φ_{p,r_i} is a Lipschitz map whose range realizes a $J_{r_i,x_0}-$holomorphic disk of B^4. Using now the result of J.Jost (theorem 3.1 page 18 of [Jo]), we deduce the existence of a map $\Psi\in W^{1,2}\cap C^{0,\alpha}(D^2,D^2)$ such that

$$\Psi^*\Phi^*_{p,r_i}g_{r,x_0}=\lambda\ dx_1\wedge dx_2$$

where $g_{r_i,x_0}:=g(r_ix+x_0)$. Thus $\Phi_p\circ\Psi$ is a $J_{r,x_0}-$holomorphic map from D^2 into (B^4,J_{r_i,x_0}). It is then a conformal and therefore a weakly harmonic map from D^2 into (B^4,g_{r_i,x_0}). We then deduce that $\Phi_p\circ\Psi$ is C^∞. We have then proved that modulo a-priori a $\mathcal{H}^2-$measure zero set $u^{-1}(p)\cap B_{r_i}$ is contained in a smooth $J-$holomorphic disk. We claim that any $x_0'\in B_{r_i}\cap\mathrm{Sing}\, u$ has to belong to this $J-$holomorphic disk. Indeed this is a simple consequence of the fact that this $J-$holomorphic disk is a closed set combined with the asymptotic analysis of u about any singular point given by proposition II.4 that tells us that $\mathrm{Sing}\, u$ is in the closure of a $\mathcal{H}^2-$significant part of $u^{-1}(p)$ for any p. Take now $p' \neq p$ being also a regular value of u different from the north pole. We have that $\mathrm{Sing}\, u$ is contained in the intersection of the two $J-$holomorphic disks produced by $u^{-1}(p)$ and $u^{-1}(p')$ in B_{r_i} that have to be not identical of course. Using again theorem 7.1 of [MW] we deduce that the number of intersection points is finite and that $\mathrm{Sing}\, u \cap B_{r_i}(x_0)$ is made of isolated points. This proves proposition III.1 for $N = 1$. ■

III.2 Proof of proposition III.1 : $N - 1 \Rightarrow N$, the need of multivalued functions.

In this subsection we assume that proposition III.1 holds for $\mathrm{Sing}^{N-1}u$: $\mathrm{Sing}^{N-1}u$ is made of isolated points in $B^4 \setminus (\mathrm{Sing}\, u \setminus \mathrm{Sing}^N u)$ and we shall prove it for $\mathrm{Sing}^N u$.

Let $x_0 \in \mathrm{Sing}^N u \setminus \mathrm{Sing}^{N-1}u$. We may always make a change of normal coordinates so that J_{x_0} becomes the standard complex structure J_0 on $\mathbb{R}^4 \simeq \mathbb{C} \oplus \mathbb{C}$ and such that

$$D_0 = \{(w, z) \quad \text{s.t.} \quad w = 0\}$$

does not belong to the family of $J_{x_0}-$holomorphic disks $D_1 \cdots D_q$ given by proposition II.7 on which u_{r,x_0} concentrates. Let p_0 be the point of S^2 equal to $\{p_0\} = u_\infty(D_0) = \phi(h(\partial D_0))$. For any regular value p of u, different from p_0, and for r small enough $(u_{r,x_0})^{-1}(p) \cap B^4_1$, that concentrates on $(u_\infty)^{-1}(p) \cup \cup_{k-1}^q D_k$ has no intersection with a cone of the form $|w| < M_{x_0}|z|$ for M_{x_0} small enough (because of the uniqueness of the tangent cone : proposition II.7). Then for any "vertical" J_0-holomorphic disk

$$D_{w_0} = \{(w, z) \quad \text{s.t.} \quad w = w_0\}$$

such that $|w_0| < \frac{1}{M_{x_0}}$, by (II.35) we have

$$(u_{r,x_0})^{-1}(p)|B^4_1 \ \cdot \ D_{w_0}|B^4_1 = N$$

On the smooth part of $(u_{r,x_0})^{-1}(p)$ in B^4_1 we consider now the map which assigns to $x \in (u_{r,x_0})^{-1}(p) \setminus \mathrm{Sing}\, u$ the J_0-holomorphic plane given by the image by Ξ, constructed in Lemma A.5 of x and the $J_{r,x_0}-$holomorphic tangent plane to $(u_{r,x_0})^{-1}(p)$ at x. Call $\xi \;:\; (u_{r,x_0})^{-1}(p) \setminus \mathrm{Sing}\, u \rightarrow \mathbb{C}P^1$ this map. In any neighborhood of $X = [0,1]$ there exists a regular value, say X_0, for ξ (i.e. $\xi^{-1}(X_0)$ is made of isolated points). Consider a small enough neighborhood and r small enough so that the $J_{r,x_0}-$holomorphic $\Theta^{X_0}_w$, for $|w| < \frac{1}{2M_{x_0}}$, keep intersecting $(u_{r,x_0})^{-1}(p)$ in B^4 with an intersection number equal to N. To simplify notations we skip X_0 by simply assuming that D_0 is a regular value for ξ : This can be made exact by slightly rotating the coordinates. Therefore we adjusted coordinates so that Θ_w is transverse to $(u_{r,x_0})^{-1}(p) \setminus \mathrm{Sing}\, u$ except along at most countably many points $\Theta(w^T_i, z^T_i)$ which are isolated in $(u_{r,x_0})^{-1}(p) \setminus \mathrm{Sing}\, u$. Using the positivity of the intersection of $J_{r,x_0}-$holomorphic curves of Micallef and White we deduce that for any w avoiding $w^T_1, w^T_2 \cdots$ and the closed set of $\mathcal{H}^2$ measure 0, $\pi(\mathrm{Sing}\, u)$ (where π is the map that assign to x the first complex coordinate of $\Theta^{-1}(x)$), Θ_w and $(u_{r,x_0})^{-1}(p)$ intersects transversally along exactly N points. Since $\mathcal{H}^2(\{w^T_1, w^T_2 \cdots\} \cup \pi(\mathrm{Sing}\, u)) = 0$, we have produced an L^∞ map from $D^2_{R_0}$, the 2-dimensional disk of radius $\frac{1}{2M_{x_0}}$ centered at x_0 and in the plane $z = 0$,into the configuration space $\mathcal{S}^N(B^4_1)$, obtained by taking the $N-$th symmetric product of B^4_1 :

$$\mathcal{S}^N(B^4_1) = B^4_1 \times B^4_1/\mathcal{S}$$

where $(x_1, \cdots, x_N) \simeq_{\mathcal{S}} (x_{\sigma(1)}, \cdots, x_{\sigma(N)})$ for any $N-$tuple $(x_1, \cdots, x_N)$ in $(B^4_1)^N$ and any permutation σ in $\mathcal{S}_N$. In other words we have shown that $\Theta^{-1} u^{-1}_{r,x_0}(p)$ is contained, modulo a set of zero $\mathcal{H}^2$ measure, into an $N-$valued graph Ψ of a map ψ from $D^2_{R_0}$ into $\mathcal{S}^N(D^2_1)$. We will use the notation $\psi(w) = \{\psi_1(w), \cdots, \psi_N(w)\}$ where the indexation is arbitrary.

From now on, in the remaining part of this subsection we pull-back the problem by means of Θ, keeping denoting the map u_{r,x_0} and the almost complex structure J_{r,x_0}

Observe that, in these new coordinates, J_{r,x_0} has the form

$$J_{r,x_0} = \begin{pmatrix} a_r & 0 \\ b_r & a'_r \end{pmatrix}$$

We will denote by $\overline{\psi}$ the single valued function into $D_1^2 \subset \mathbb{C}$ given by the average of the ψ_i

$$\overline{\psi}(w) = \frac{\displaystyle\sum_{i=1}^{N} \psi_i(w)}{N} \tag{III.1}$$

We also keep denoting π the projection along the first complex coordinate.

Since $\operatorname{Sing} u$ is closed in $\overline{u^{-1}(p)}$ and since $\mathcal{H}^2(\operatorname{Sing} u) = 0$, it is clear that $\pi(\operatorname{Sing} u)$ is a closed measure zero subset of $D^2_{R_0}$. Denote by $U = D^2_{R_0} \setminus \pi(\operatorname{Sing} u)$ the open dense subset of $D^2_{R_0}$ above which $u^{-1}_{r,x_0}(p)$ is smooth so that, for any w_0 in U, $D_{w_0} \cap u^{-1}_{r,x_0}(p)$ is made of N smooth points of $u^{-1}_{r,x_0}(p)$.

If we keep denoting (w_i^T, z_i^T), the at most countable set of points at which D_w and $U = u^{-1}_{r,x_0}(p) \setminus \operatorname{Sing} u$ are not transverse to each other (which means in particular that w_i^T is the set of points in U where ψ is not smooth). Let denote $\tilde{U} = U \setminus \{w_1^T, w_2^T \cdots\}$. $\tilde{U}$ is also an open set : indeed let $w_0 \in U$ and $\rho > 0$ such that $\overline{B_\rho(w_0)} \subset U$. On $\pi^{-1}(\overline{B_\rho(w_0)}) \cap B_1^4$, which is compact, u_{r,x_0} is smooth and for a regular p, $u^{-1}_{r,x_0}(p)$ is a compact smooth submanifold. Therefore, since we have chosen X_0 to be a regular value for ξ defined on this smooth submanifold, $\xi^{-1}(X_0) \cap \pi^{-1}(\overline{B_\rho(w_0)})$ is made of finitely many points and we deduce that the w_i^T are isolated in $B_\rho(w_0)$ which implies that once we remove the w_i^T to U we still get an open subset of $D^2_{R_0}$.

Since D_w intersects the smooth part of $u^{-1}_{r,x_0}(p)$ transversally for each w in $\tilde{U}$, for such a w there exists a small neighborhood Δ of it above which ψ splits as a standard single-valued map into $(D_1^2)^N$ and the indexation makes sense locally. This permits to compute derivatives of the ψ_is and the multivalued functions such as $\{\frac{\partial \psi_i}{\partial w}\}$, $\{\frac{\partial \psi_i}{\partial \overline{w}}\}$, $\nabla \psi_i$... are well defined on $\tilde{U}$ and we have the following lemma :

Lemma III.1 *Let $\overline{\psi}$ defined by (III.1). We have that $\overline{\psi} \in W^{1,2}(U, D_1^2)$, moreover there exists C independent on r and p such that*

$$\int_U |\nabla \overline{\psi}|^2 \leq C \mathcal{H}^2 \left(u^{-1}_{r,x_0}(p) \cap B_1^4 \right) \tag{III.2}$$

We also have

$$\int_{\tilde{U}} \sum_{i=1}^{N} |\nabla \psi_i|^2 \leq C \mathcal{H}^2 \left(u^{-1}_{r,x_0}(p) \cap B_1^4 \right) \tag{III.3}$$

Proof of the lemma III.1.

If we prove that $\overline{\psi}$ is $W^{1,2}$ in U apart from the w_i^T, because we have at most countably many points which are isolated in U (as we will show it right below) and $\overline{\psi}$ is L^∞, we can deduce that $\overline{\psi}$ is $W^{1,2}$ in the whole of U (see the detail of the argument in the proof of lemma III.3). To simplify the presentation of the computations we reparametrize each vertical disc D_w using a diffeomorphism $\mathcal{Z}\ :\ (w,z)\ \to (w,\zeta_w(z))$ so that, for any $w\in D^2_{R_0}$ the map $z\to (w,\zeta_w(z))$ is $J_{r,x_0}-$holomorphic. If we keep denoting again u_{r,x_0}, the composition of u_{r,x_0} with this diffeomorphism $\mathcal{Z}$ and if we also keep denoting J_{r,x_0} the pull-back of J_{r,x_0} by this diffeomorphism : $d\mathcal{Z}^{-1}J_{r,x_0}d\mathcal{Z}$, J_{r,x_0} now writes in these new coordinates

$$J_{r,x_0}=\left(\begin{array}{cc} a_r & 0\\ & \\ b_r & \left(\begin{array}{cc} 0 & -1\\ 1 & 0\end{array}\right)\end{array}\right) \tag{III.4}$$

where

$$\left\|a_r-\left(\begin{array}{cc} 0 & -1\\ 1 & 0\end{array}\right)\right\|_{C^l(B^4_1)}\longrightarrow 0\quad \text{ as } r\to 0 \tag{III.5}$$

and $\|b_r\|_{C^l(B^4_1)}\longrightarrow 0$ for any $l\in\mathbb{N}$.

We have the following lemma :

Lemma III.2 *The multivalued functions $\{\psi_i\}$, $\{\frac{\partial\psi_i}{\partial w}\}$ and $\{\frac{\partial\psi_i}{\partial\overline{w}}\}$ are related as follows on $\tilde{U}$:*

$$\text{For } i=1\cdots N\quad \frac{\partial\psi_i}{\partial\overline{w}}=\delta_r(\psi_i)\ \frac{\partial\psi_i}{\partial w}+\nu_r(\psi_i) \tag{III.6}$$

where δ_r and ν_r are smooth function depending on r satisfying for any $l\in\mathbb{N}$

$$\|\delta_r\|_{C^l}+\|\nu_r\|_{C^l}\to 0$$

as r tends to 0.

Proof of lemma III.2 : Since $u^{-1}_{r,x_0}(p)\setminus\text{Sing}\, u$ is a $J_{r,x_0}-$holomorphic surface the $\{(w,\psi_i(w)) \quad \text{s.t. } w\in\Delta\}$ are $J_{r,x_0}-$holomorphic graphs. This writes

$$\left\{\begin{array}{l} J_{r,x_0}\left(\begin{array}{c}1\\ 0\\ \dfrac{\partial\psi_i}{\partial x_1}\end{array}\right)=\lambda_1\left(\begin{array}{c}1\\ 0\\ \dfrac{\partial\psi_i}{\partial x_1}\end{array}\right)+\mu_1\left(\begin{array}{c}0\\ 1\\ \dfrac{\partial\psi_i}{\partial x_2}\end{array}\right)\\ J_{r,x_0}\left(\begin{array}{c}0\\ 1\\ \dfrac{\partial\psi_i}{\partial x_2}\end{array}\right)=\lambda_2\left(\begin{array}{c}1\\ 0\\ \dfrac{\partial\psi_i}{\partial x_1}\end{array}\right)+\mu_2\left(\begin{array}{c}0\\ 1\\ \dfrac{\partial\psi_i}{\partial x_2}\end{array}\right)\end{array}\right. \tag{III.7}$$

where the 2×2 matrix

$$\left(\begin{array}{cc}\lambda_1 & \mu_1\\ \lambda_2 & \mu_2\end{array}\right)=a_r$$

and a_r is the matrix in (III.4) satisfying (III.5). If we denote

$$b_r:=\left(\begin{array}{cc}b_{11} & b_{12}\\ b_{21} & b_{22}\end{array}\right)$$

(III.7) writes

$$\begin{aligned}\left[1-\frac{1}{4}\left((\mu_1+1)+(1-\lambda_2)\right)+\frac{i}{4}(\lambda_1+\mu_2)\right]\ \frac{\partial\psi_i}{\partial\overline{w}}\\ =\left[\frac{1}{4}\left((1+\mu_1)+(\lambda_2-1)\right)+\frac{i}{4}(\mu_2-\lambda_1)\right]\ \frac{\partial\psi_i}{\partial w}\\ +\frac{1}{2}(b_{21}+b_{12})+\frac{i}{2}(b_{11}-b_{22})\end{aligned} \tag{III.8}$$

This yields the desired result and Lemma III.2 is proved. $\blacksquare$

Given w in $\tilde{U}$, we work, now on a neighborhood Δ of w in $\tilde{U}$ above which $\{\psi_i\}$ splits into single-valued functions into $(D^2_1)^N$. From Lemma III.2 we deduce

$$\left|\frac{\partial\psi_i}{\partial\overline{w}}\right|\le\delta(r)\left[\left|\frac{\partial\psi_i}{\partial w}\right|+1\right] \tag{III.9}$$

where $\delta(r) \to 0$ as $r \to 0$. And therefore

$$2|\nabla\psi_i|^2 = \left|\frac{\partial\psi_i}{\partial w}\right|^2 + \left|\frac{\partial\psi_i}{\partial \overline{w}}\right|^2 \le (1+2\delta^2(r))\left|\frac{\partial\psi_i}{\partial w}\right|^2 + 2\delta^2(r) \qquad \text{(III.10)}$$

Since u_{r,x_0} is a $J_{r,x_0}-$holomorphic map into $\mathbb{C}P^1$, we have for $k = 1\cdots 4$

$$j_0\frac{\partial u_{r,x_0}}{\partial x_k} = du_{r,x_0} \quad \cdot \quad J_{r,x_0}\frac{\partial}{\partial x_k}$$

using the fact that $J_{r,x_0}\frac{\partial}{\partial x_3} = \frac{\partial}{\partial x_4}$ and that $J_{r,x_0}\frac{\partial}{\partial x_4} = -\frac{\partial}{\partial x_3}$, we have, using the complex coordinates on $\mathbb{R}^4 \simeq \mathbb{C}\oplus\mathbb{C}$,

$$\frac{\partial u_{r,x_0}}{\partial \overline{z}} = 0$$

Using also that $u_{r,x_0}(w,\psi_i(w)) \equiv p$ we deduce

$$\frac{\partial\psi_i}{\partial w} = \frac{\dfrac{\partial u_{r,x_0}}{\partial w}}{\dfrac{\partial u_{r,x_0}}{\partial z}}(w,\psi_i(w)) \qquad \text{(III.11)}$$

combining this fact with (III.10), we get

$$\begin{aligned}\int_\Delta \sum_{i=1}^N |\nabla\psi_i|^2 \quad &\le \sum_{i=1}^N C\int_\Delta \left|\frac{\partial_w u_{r,x_0}}{\partial_z u_{r,x_0}}\right|^2 (w,\psi_i(w)) + 1 \\ &\le \sum_{i=1}^N C\int_{\Psi_i^{-1}(\Delta)} \left[\left|\frac{\partial_w u_{r,x_0}}{\partial_z u_{r,x_0}}\right|^2 + 1\right] \frac{1}{|\Psi_i^*\Omega_{r,x_0}|}\; d\mathcal{H}^2\end{aligned} \qquad \text{(III.12)}$$

where $\Psi_i(w) = (w,\psi_i(w))$ and Ω_{r,x_0} is the almost-Kähler form compatible with J_{r,x_0} and the metric $g_{r,x_0} = g(rx+x_0)$. We have that Ω_{r,x_0} converges to $dw\wedge d\overline{w} + dz\wedge d\overline{z}$ as r tends to zero, therefore, in view also of (III.9), $|\Psi_i^*\Omega_{r,x_0}| \simeq 1 + \left|\frac{\partial\psi_i}{\partial w}\right|^2$. Using (III.11) again, (III.12) becomes

$$\begin{aligned}\int_\Delta \sum_{i=1}^N |\nabla\psi_i|^2 \quad &\le \int_{\pi^{-1}(\Delta)\cap u_{r,x_0}^{-1}(p)} \left[\left|\frac{\partial_w u_{r,x_0}}{\partial_z u_{r,x_0}}\right|^2 + 1\right] \frac{1}{1+\left|\dfrac{\partial_w u_{r,x_0}}{\partial_z u_{r,x_0}}\right|^2} \\ &\le C\mathcal{H}^2(u_{r,x_0}^{-1}(p)\cap\pi^{-1}(\Delta))\end{aligned}$$

This proves lemma III.1. ■

Remark III.1 *Although U is an open dense set of $D^2_{R_0}$, it is not straightforward at all to extend lemma III.1 ($\overline{\psi} \in W^{1,2}(U, D^2_1)$) to the whole of $D^2_{R_0}$ even knowing that $\overline{\psi} \in C^0(U, D^2_1)$ (this is not correct for arbitrary functions): distributional derivatives of $\overline{\psi}$ may a-priori contain singular parts on $D^2_{R_0} \setminus U$ (which is a closed measure 0 set). This part of the proof is missing in Taubes' work (see page 76 of [Ta]). The passage from $W^{1,2}(U)$ to $W^{1,2}(D^2_{R_0})$ could only be asserted if some a-priori better control on the size of $D^2_{R_0} \setminus U$ would be known (such as $\mathcal{H}^1(D^2_{R_0} \setminus U) < +\infty$, for instance) which means some a-priori better control on the size of the singular set of a $J-$holomorphic rectifiable cycles (but this was the purpose of [Ta]).*

We present below a proof of this fact. Precisely we have

Lemma III.3 *Under the above notations, we have that the unique valued function $\overline{\psi} := \frac{\sum_{i=1}^N \psi_i}{N}$ is in $W^{1,2}(D^2_{R_0}, D^2_1)$ moreover*

$$\int_{D^2_{R_0}} |\nabla \overline{\psi}|^2 \le C\mathcal{H}^2 \left(u^{-1}_{r,x_0}(p) \cap B^4_1\right) \tag{III.13}$$

.

Before proving lemma III.3, we need to split $\text{Sing}^N u \setminus \text{Sing}^{N-1} u$ into two parts. Denote $\text{Sing}^N_0 u$ the set of points x_0 in $\text{Sing}^N u \setminus \text{Sing}^{N-1} u$ such that the family of Radon measures $|\nabla u_{r,x_0}|^2 \, dx$ does <u>not</u> concentrate on a <u>single</u> $J_{x_0}-$holomorphic disk with a multiplicity N. $\text{Sing}^N_0 u$ is the part of $\text{Sing}^N u \setminus \text{Sing}^{N-1} u$ that can be handled relatively easily : we have the following lemma.

Lemma III.4 *For any $x_0 \in \text{Sing}^N_0 u$, there exists $r_0 > 0$ such that*

$$\left(\text{Sing}^N u \setminus \text{Sing}^{N-1} u\right) \cap B^4_{r_0}(x_0) = \{x_0\}$$

Proof of lemma III.4 : By contradiction. Assume that there exists $x^n \in \text{Sing}^N u \setminus \text{Sing}^{N-1} u$ such that $x^n \to x_0$ as $n \to +\infty$. Choose $r_n = 2|x^n - x_0|$ and denote $x^n_{r_n} = x^n / r_n$. From proposition II.4 we can extract a subsequence that we keep denoting x^n and r_n such that

$$u_{r_n,x_0} \rightharpoonup u_\infty \qquad \text{in } W^{1,2}$$

and

$$|\nabla u_{r_n,x_0}| \, dx \rightharpoonup |\nabla u_\infty|^2 \, dx + \sum_{k=1}^q N_k \, \mathcal{H}^2 \llcorner D_k$$

as Radon measures. Using proposition II.7 (the almost uniqueness of the cone) and the assumption that $x_0 \in \mathrm{Sing}_0^N u$ we know that either $u_\infty \neq$constant or $q > 1$. Observe in particular that in both cases we have $N_k < N$ for any k.

On $B^4_1 \setminus B^4_{1/4}$, for any $\varepsilon > 0$ and r_n small enough, we have

$$\mathrm{Sing}\, u_{r_n,x_0} \subset \cup_{k=1}^q T_k^\varepsilon = \cup_{k=1}^q \left\{ x \in B^4_1 \setminus B^4_{1/4} \quad \text{s.t.} \quad \mathrm{dist}(x, D_k) \leq \varepsilon \right\}$$

Let k_0 such that $x^n_{r_n} \in T^\varepsilon_{k_0}$. Arguing like in the proof of proposition II.7, using in particular (II.35), it is easy to produce a $J_{r_n,x_0}-$holomorphic disk passing through $x^n_{r_n}$ transverse to D_{k_0} (with an angle bounded from below by a positive constant) whose intersection with $u^{-1}_{r_n,x_0}(p) \cap T^\varepsilon_{k_0}$, for a generic p in $\mathbb{C}P^1$, is at least N. This would contradict that $N_{k_0} < N$. Lemma III.4 is proved. ■

Proof of lemma III.3 in the case where $x_0 \in \mathbf{Sing}_0^N u$:

We are assuming that proposition III.1 holds for the integers less or equal to $N-1$. Then because of the previous lemma III.4, we have that $\mathrm{Sing}\ u_{r,x_0} \cap B^4_1$, for r small enough, is made of at most countably many points with only one possible accumulation point which is x_0. Therefore $D^2_{R_0} \setminus U$ is also made of isolated points in $D^2_{R_0} \setminus \{0\}$ plus 0. Since $\overline{\psi} \in L^\infty \cap W^{1,2}(U, D^2_1)$ we can deduce that $\overline{\psi} \in W^{1,2}(D^2_{R_0}, D^2_1)$ as follows. Let $\varepsilon > 0$. Since the singularities are isolated except perhaps 0 we may choose finitely many w in $D^2_{R_0}$, say $w_0, w_1 \cdots w_Q$ with $w_0 = 0$ and finitely many radii $\rho_0 \cdots \rho_Q$ such that $\sum_i \rho_i < \varepsilon$ and $D^2_{R_0} \setminus U \subset \cup_i B^2_{\rho_i}(w_i)$. For any test function $\varphi \in C^\infty_0(D^2_{R_0})$ we have for $l = 1, 2$

$$\begin{aligned}
\left| \int_{D^2_{R_0}} \overline{\psi}\, \frac{\partial \varphi}{\partial x_l} \right| &\leq \left| \int_{D^2_{R_0} \setminus \cup_i B_{\rho_i}(w_i)} \varphi\, \frac{\partial \overline{\psi}}{\partial x_l} \right| \\
&\quad + \left| \int_{\cup_i B_{\rho_i}(w_i)} \overline{\psi}\, \frac{\partial \varphi}{\partial x_l} \right| + \sum_i \left| \int_{\partial B_{\rho_i}(w_i)} \overline{\psi}\ \varphi\ e_l \cdot \nu \right| \\
&\leq \|\overline{\psi}\|_{W^{1,2}(U)}\ \|\varphi\|_{L^2(D^2)} + C\varepsilon\ \left[\|\varphi\|_\infty + \|\nabla \varphi\|_\infty \right]\ \|\overline{\psi}\|_\infty
\end{aligned}$$

making ε tend to zero we get the desired result and lemma III.3 is proved in the case where $x_0 \in \mathrm{Sing}_0^N u$. ■

Denote by $\mathrm{Sing}^N_\star u$ the complement of $\mathrm{Sing}^N_0 u$ in $\mathrm{Sing}^N u \setminus \mathrm{Sing}^{N-1} u$ (this is a closed set) : this is the set of points x_0 in $\mathrm{Sing}\, u$ such that

$$|\nabla u_{r,x_0}|^2\ dx \to 8\pi\, N\ \mathcal{H}^2 \llcorner D_{x_0}$$

as $r \to 0$ for some $J_{x_0}-$holomorphic disk D_{x_0}. It remains to prove Lemma III.3 for $x_0 \in \mathrm{Sing}^N_\star u$ but before to do so we still need the following lemma.

Lemma III.5 *Let $x_0 \in Sing^N_\star u$. For any $\varepsilon > 0$ there exists $r > 0$ such that for any $y \in Sing^N_\star u \cap B_r(x_0)$*

$$|(\widehat{D_y; D_{x_0}})| \le \varepsilon$$

where $(\widehat{D_y; D_{x_0}})$ denotes the angle between the 2 planes defined as $n_y \wedge n_{x_0}$ where n_y and n_{x_0} are respectively the unit two-covector of the two J_y and $J_{x_0}-$holomorphic planes D_y and D_{x_0} oriented by J_y and J_{x_0}.

Proof of lemma III.5 : We argue by contradiction. Assume there exists $y^n \to x_0$ such that $y^n \in \mathrm{Sing}^N_\star u$ and $|(\widehat{D_y; D_{x_0}})| \ge \alpha > 0$. Consider $r^n = 2|y^n - x_0|$. Modulo a change of coordinate we may assume that J_{x_0} is the standard complex structure J_0 on $\mathbb{R}^4 \simeq \mathbb{C}\oplus\mathbb{C}$ and that D_{x_0} is the holomorphic "horizontal" flat plane $D_0 = \{(w, 0) \quad w \in \mathbb{C}\}$. Denote

$$D_0^\varepsilon := \{(w, z) \quad |z| \le \varepsilon\}$$

From r_n we may extract a subsequence, still denoted r_n, such that u_{r_n,x_0} weakly converges to a constant map u_∞ that we may assume to be the north pole (to fix ideas). It converges strongly in $C^l_{loc}(B^4_1 \setminus D_0)$ for any $l \in \mathbb{N}$. Therefore for any $\varepsilon > 0$ and a.e. $p \in \mathbb{C}P^1$ there exists $n_0 \in \mathbb{N}$ such that for $n \ge n_0$ $u^{-1}_{r_n,x_0}(p) \cap B^4_1 \subset D^0_\varepsilon$. Denote $z^n = \frac{y^n - x_0}{2|y^n - x_0|}$, we have $z_n \in \mathrm{Sing}^N_\star u_{r_n,x_0}$ and D_{z_n} makes an angle larger that α with D_0.

We are going to exhibit a J_{r_n,x_0} holomorphic disk $\tilde{D}_{z_n}$ in B^4_1 verifying

i) $z_n \in \tilde{D}_{z_n}$

ii) $T_{z_n}\tilde{D}_{z_n} \ne D_{z_n}$

iii) $\|\tilde{D}_{z_n}\|_{C^2} \le C$ indep. of n and $|(\widehat{T_{z_n}\tilde{D}_{z_n}; D_0})| \ge \alpha/2$.

iv) $\tilde{D}_{z_n}$ intersects $u^{-1}_{r_n,x_0}(p)\cap B^4_1\setminus \mathrm{Sing}\, u$ transversally and the intersection is not empty.

($\|\tilde{D}_{z_n}\|_{C^2}\le C$ means that $\tilde{D}_{z_n}$ is the graph of a holomorphic map into (B^4_1, J_{r_n,x_0}) whose C^2 norm is uniformly bounded). Assume we are able to construct such a disk. Condition ii) ensures, because of (II.35), that, for any ρ smaller than some ρ^0_n,

$$< [B_\rho(z_n)], u_{r_n,x_0}, p > \ \cdot [\tilde{D}_{z_n}]\llcorner B_\rho(z_n) = N \tag{III.14}$$

Condition iv) will ensure that for ρ small enough, using the positivity of the intersection of $J_{r_n,x_0}-$holomorphic disks of Micallef and White

$$< \left[B^4_1\setminus B_\rho(z_n)\right], u_{r_n,x_0}, p > \ \cdot [\tilde{D}_{z_n}]\llcorner \left(B^4_1\setminus B_\rho(z_n)\right)\ge 1 \tag{III.15}$$

Combining (III.14) and (III.15) we would get, using again the positivity of the intersection of $J_{r_n,x_0}-$holomorphic disks of Micallef and White

$$< [B^4_1], u_{r_n,x_0}, p > \ \cdot [\tilde{D}_{z_n}]\llcorner B^4_1 > N \tag{III.16}$$

But condition iii) implies that $\tilde{D}_{z_n}$ is transverse to D^ε_0 (it intersects D^ε_0 along a single disk transverse to D_0 in B^4_1) and (III.16) would contradicts that the degree of u_{r_n,x_0} on such a disk has to be N (i.e. $|\nabla u_{r_n,x_0}|^2\, dx\to 8\pi\, N\, \mathcal{H}^2\llcorner D_0$).

Construction of $\tilde{D}_{z_n}$ satisfying i)...iv)

For X being a flat $J_{z_n}-$holomorphic disk passing through z_n and close enough to D_{z_n} we consider the $J_{r_n,x_0}-$holomorphic disk $D_{z_n,X}$ given by Lemma A.2. The family of these disks, for X varying in a neighborhood V of D_{z_n} in $\mathbb{C}P^1$ (recall that the manifold of $J_{z_n}-$holomorphic flat disks identifies with $\mathbb{C}P^1$), foliates a subdomain of B^4_1 that contains the cone, C_{z_n}, made of the $J_{z_n}-$holomorphic flat disks whose angle with D_{z_n} is less than $1/2$ say (last condition of Lemma A.2).

We claim that for every $\rho>0$ there exists a non zero measure of $X\in V$ such that

$$a)\qquad D_{X,z_n}\cap \mathrm{Sing}\, u\cap B_\rho(z_n)=\{z_n\}$$

$$b)\qquad \mathcal{H}^2\left(u_{r_n,x_0}(B_\rho(z_n)\cap D_{X,z_n})\right)>0$$

Clearly, since $\mathcal{H}^2(\mathrm{Sing}\, u) = 0$ and since D_{X,z_n} realizes a foliation of a cone parametrized by a non empty open subset V of $\mathbb{C}P^1$, a simple integral geometry argument gives that there exists a full measure of X in V such that $a)$ holds.

Take one of the X such that $b)$ does not hold and such that $a)$ holds. If there would exists $x \in D_{X,z_n} \cap B_\rho(z_n)$ such that $d(u_{r_n,x_0}|D_{X,z_n}) \neq 0$ ($u_{r_n,x_0}|D_{X,z_n}$ means the restriction of u_{r_n,x_0} to D_{X,z_n}), in other words if there exists $e \in T_x D_{X,z_n}$ verifying $du_{r_n,x_0} \cdot e \neq 0$, then, we would have also $du_{r_n,x_0} \cdot J_{r_n,x_0} e \neq 0$. But since $J_{r_n,x_0} e \in T_x D_{X,z_n}$ (D_{X,z_n} is $J_{r_n,x_0}-$holomorphic) this would imply that the rank of $d(u_{r_n,x_0}|D_{X,z_n})$ at x is 2 and this would contradicts the fact that for the X we consider $a)$ holds but $b)$ does not hold.

Assume now that there exists $\rho > 0$ such that

$$\mathcal{H}^2\left(u_{r_n,x_0}(B_\rho(z_n) \cap D_{X,z_n})\right) = 0 \qquad \text{for almost every } X \in V.$$

Because of the previous remarks we have then established that for almost every X in V, u_{r_n,x_0} is constant on $B_\rho(z_n) \cap D_{X,z_n}$. Since, from proposition II.4 and the fact that z_n is in $\mathrm{Sing}^N_\star u$, there exists $\rho_i \to 0$ for which $u_{r_n,x_0}(\rho_i x + z_n) \to u_\infty$ =constant= p_0 strongly in $C^k(B^4_1 \setminus D_{X,z_n})$. This implies that the constant is the same, equal to p_0 for $\mathcal{H}^2-$almost all the X above :

$$\text{for } \mathcal{H}^2 \text{ a.e. } X \in V \quad \forall x \in D_{X,z_n} \cap B_\rho(z_n) \quad u_{r_n,x_0}(x) = p_0$$

Thus we would get that u_{r_n,x_0} is constant equal to p_0 inside the portion of cone $C_{z_n} \cap B_\rho(z_n)$. This is in contradiction with the fact that $|\nabla u_{r_n,x_0}(\rho_i x + z_n)|^2\ dx \to 8\pi\, N\ \mathcal{H}^2 \llcorner D_{z_n}$ as $\rho_i \to 0$ since D_{z_n} is the "center" of C_{z_n}. Then we have proved that $a)$ and $b)$ hold for a non zero $\mathcal{H}^2-$measure of X in V.

Take now X_1 such that $a)$ and $b)$ hold and $x_1 \in D_{X_1,z_n} \cap B_\rho(z_n)$ such that the rank of $d(u_{r_n,x_0}|D_{X_1,z_n})$ at x_1 is 2. Then, there exists a geodesic ball $B_\sigma(u_{r_n,x_0}(x_1))$ in $\mathbb{C}P^1$ such that, for an open neighborhood W_{x_1} of x_1 in $D_{X_1,z_n} \cap B_\rho(z_n)$ $u_{r_n,x_0} : W_{x_1} \to B_\sigma(u_{r_n,x_0}(x_1))$ realizes a diffeomorphism. Let $p_1 \in B_\sigma(u_{r_n,x_0}(x_1))$ be a regular value for u_{r_n,x_0} in B^4_1. For a.e. X in V $u^{-1}_{r_n,x_0}(p_1)$ is transverse to D_{X,z_n}. We may always assume that we have chosen X_1 to be a point of density 1 in the subset of $X \in V$ for which $a)$ and $b)$ hold true. Therefore we may find an X_2 as close as X_1 as we want, satisfying $a)$ and $b)$ and such that $u^{-1}_{r_n,x_0}(p_1) \cap B^4_1 \setminus \mathrm{Sing}\, u$ is transverse to D_{X_2,z_n}. Moreover $u^{-1}_{r_n,x_0}(p_1) \cap D_{X_2,z_n} \cap B_\rho(z_n) \neq 0$. It is now easy to check that $\tilde{D}_{z_n} := D_{X_2,z_n}$ fulfills i)...iv). Lemma III.5 is proved. ∎

We need a last lemma before proving lemma III.3 in the case where $x_0 \in \mathrm{Sing}^N_\star u$.

Lemma III.6 *Let $x_0 \in Sing^N_\star u$. Use normal coordinates such that J_{x_0} becomes the standard complex structure on $\mathbb{R}^4 \simeq \mathbb{C} \oplus \mathbb{C}$ and such that the concentration $J_{x_0}-$holomorphic disk D_{x_0} coincides with $D_0 = \{(w,z) \quad s.t.\ z = 0\}$. Let $\overline{\psi} = \frac{1}{N}\sum_{i=1}^N \psi_i$ be the average of the $N-$valued function $\psi\ :\ D^2_{R_0} \to \mathcal{S}^N(D^2_1)$ constructed above, whose graph contains $u^{-1}_{r,x_0}(p) \cap D^2_{R_0} \times D^2_1$ for a regular value p, modulo a zero $\mathcal{H}^2$-measure set. There exists a representative of $\overline{\psi}$ such that, for r small enough, we have*

$$
\begin{aligned}
&\forall w_1 \in \pi(Sing^N_\star u_{r,x_0}) \cap D^2_{R_0/2} \qquad \text{for a. e. } w_2 \in D^2_{R_0/2}\\
&\qquad\qquad |\overline{\psi}(w_1) - \overline{\psi}(w_2)| \le |w_1 - w_2|
\end{aligned}
\tag{III.17}
$$

The constant 1 as a "semi-Lipschitz" coefficient for $\overline{\psi}$ in (III.17) is arbitrary and could be replaced by any fixed positive number modulo a choice of r which has to be taken smaller for a choice of a smaller positive coefficient.

Proof of Lemma III.6 : Let r be the radius given by lemma III.5 for $\varepsilon = \frac{1}{2}$. Pick $w_1 \in \pi(\mathrm{Sing}^N_\star u_{r,x_0}) \cap D^2_{R_0/2}$. Clearly, there can be only a single $y_1 = (w_1, z_1)$ in $\pi^{-1}(w_1) \cap [\mathrm{Sing}\, u \cup u^{-1}_{r,x_0}(p)]$ otherwise the intersection number of $u^{-1}_{r,x_0}(p)|B^4_1 \cdot D_{w_1}|B^4_1 > N$. Take a representative of $\overline{\psi}$ such that $\overline{\psi}(w_1) := z_1$ (this is possible since $\mathcal{H}^2(\pi(\mathrm{Sing}^N_\star u_{r,x_0})) = 0$. We will check that for any $w_2 \in D^2_{R_0/2} \setminus \pi(\mathrm{Sing}\, u)$

$$
|\overline{\psi}(w_1) - \overline{\psi}(w_2)| \le |w_1 - w_2|
$$

Assume this would not be the case, then we would find $y_2 = (w_2, z_2) \in \pi^{-1}(w_2) \cap u^{-1}_{r,x_0}(p) \setminus \mathrm{Sing}\, u$ such that $|z_1 - z_2| > |w_1 - w_2|$. Introduce then the $J_{r,x_0}-$holomorphic disk D_{y_1,y_2} passing trough y_1 and y_2 given by Lemma A.3 that can be chosen as close as we want to the holomorphic disk

$$
D^0_{y_1,y_2} = \left\{ \left(w_2 + (z - z_2)\frac{w_1 - w_2}{z_1 - z_2}, z \right) \right\}
$$

for r small enough (in order to ensure that $\|J_{r,x_0} - J_0\|_{C^l}$ is as small as we need for any given $l \in \mathbb{N}$). Because of lemma III.5, $D^0_{y_1,y_2}$ is clearly transverse to the concentration disk D_{y_1} at y_1 and since $T_{y_1}D_{y_1,y_2}$ can be taken very close to $T_{y_1}D^0_{y_1,y_2}$ (for r small enough) we have that D_{y_1,y_2} is also transverse to D_{y_1}. Therefore, using (II.35), we have for ρ small enough,

$$< [B_\rho(y_1)], u_{r,x_0}, p > \ \cdot [D_{y_1,y_2}] \llcorner B_\rho(y_1) = N \tag{III.18}$$

but since $y_2 \in D_{y_1,y_2} \cap u^{-1}_{r,x_0}(p) \setminus \mathrm{Sing}\, u$, we also have

$$< [B^4_1 \setminus B_\rho(y_1)], u_{r,x_0}, p > \ \cdot [D_{y_1,y_2}] \llcorner B^4_1 \setminus B_\rho(y_1) > 0 \tag{III.19}$$

Combining (III.18) and (III.19) we have

$$< [B^4_1], u_{r,x_0}, p > \ \cdot [D_{y_1,y_2}] \llcorner B^4_1 > N$$

Since D_{y_1,y_2} is transverse to the "horizontal disk" D_0, the last inequality contradicts that $N_{x_0} = N$. Therefore Lemma III.6 is proven. ■

Proof of Lemma III.3 for $x_0 \in \mathbf{Sing}^N_\star u$. Using arguments like the one we developed to prove Lemma III.3 in the case where $x_0 \in \mathrm{Sing}^N_0 u$, since $\mathrm{Sing}^N_0 u$ is made of at most countably many points, lemma III.1 extends replacing U by the complement of $\pi(\mathrm{Sing}^N_\star u)$.

Let $\delta > 0$. From the family $\{B_\delta(w)\}$ where w varies in $\pi(\mathrm{Sing}^N_\star u_{r,x_0})$ we extract a Besicovitch covering $\{B_\delta(w_i)\}_{i=1\cdots K}$ so that any point in $D^2_{R_0}$ is covered by at most $n^\star$ balls ($n^\star$ is a universal number arising in the Besicovitch construction). Call V^j_δ, $j = 1\cdots Q$, the connected components of $\cup_i B_\delta(w_i) \setminus \cup_i \partial B_\delta(w_i)$. To each V^j_δ we assign a $i_j \in \{1\cdots K\}$ such that $\partial B_\delta(w_{i_j}) \cap \partial V^j_\delta \neq \emptyset$. We have in particular that $\sup\{|w - w_{i_j}|$ s.t. $w \in V^j_\delta\} \le \delta$. Denote $V_\delta = \cup_j V^j_\delta$. Let $\varphi \in C^\infty_0(D^2_{R_0})$ be a test function. For any $l = 1, 2$ we have, using integration by parts,

$$\begin{aligned}\int_{D^2} \frac{\partial\varphi}{\partial x_l}\, \overline{\psi} \ &= -\int_{D^2\setminus V_\delta} \frac{\partial\overline{\psi}}{\partial x_l}\, \varphi + \sum_j \int_{\partial V^j_\delta} \varphi\ [\overline{\psi} - \overline{\psi}(w_{i_j})]\ \nu_l \\ &\quad + \sum_j \overline{\psi}(w_{i_j}) \int_{V^j_\delta} \frac{\partial\varphi}{\partial x_l} + \int_{V_\delta} \frac{\partial\varphi}{\partial x_l}\, \overline{\psi}\end{aligned} \tag{III.20}$$

where ν_l is the projection of the unit normal vector to ∂V^j_δ on the $l-$th direction. Since $\sup\{|w - w_{i_j}|$ s.t. $w \in V^j_\delta\} \le \delta$, using that $w_{i_j} \in \pi(\mathrm{Sing}^N_\star u_{r,x_0})$

and Lemma III.6, we deduce

$$\begin{aligned}\left|\int_{D^2}\frac{\partial\varphi}{\partial x_l}\,\overline{\psi}\right| &\le \|\varphi\|_{L^2(D^2)}\|\nabla\overline{\psi}\|_{L^2(U)}\\ &+\delta\ \|\varphi\|_\infty\ \sum_j|\partial V_\delta^j|+C\int_{V_\delta}|\nabla\varphi|\end{aligned}\tag{III.21}$$

(observe that we could have to slightly modify each V_δ^j to ensure, modulo some Fubini type argument, that $\mathcal{H}^1-$a.e $w\in\partial V_\delta^j$ is in $D^2_{R_0}\setminus\pi(Sing\,u)$ in order to be able to apply (III.17)). Observe that we have

$$\begin{aligned}&\delta\sum_j|\partial V_\delta^j|=2\sum_i\delta|\partial B_\delta(w_i)|=\frac{4}{\pi}\sum_i|B_\delta(w_i)|\\ &\le n^\star\frac{4}{\pi}|\{x\quad;\quad dist(x;D^2\setminus U)\le\delta\}\end{aligned}\tag{III.22}$$

Using the fact that $D^2\setminus U$ is measure zero closed set, we have

$$\left|\left\{x\quad;\quad dist(x;D^2\setminus U)\le\delta\right\}\right|\to 0\tag{III.23}$$

as $\delta\to 0$. Combining (III.22) and (III.23) we deduce that the two last terms in the right-hand-side of (III.21) tends to zero as δ tend to 0 and we finally obtain that for any $\varphi\in C_0^\infty(D^2)$,

$$\left|\int_{D^2}\overline{\psi}\ \nabla\varphi\right|\le C\|\varphi\|_{L^2(D^2)}\|\nabla\overline{\psi}\|_{L^2(U)}$$

Therefore the pointwise derivative of $\overline{\psi}$ on $D^2_{R_0}$ coincide with the distributional derivative of $\overline{\psi}$ on $D^2_{R_0}$ and $\overline{\psi}\in W^{1,2}(D^2_{R_0})$. We have proved lemma III.3 in any cases. ∎

The multivalued function $\{\psi_i\}$ solves a "PDE" (relation (III.6)) on a subset $\tilde{U}$ of $D^2_{R_0}$. This "PDE" is a soft almost linear perturbation of the $\overline{\partial}$-equation $\overline{\partial}f=0$. The difficulty arising from the fact that the "PDE" is not solved on the whole $D^2_{R_0}$, but only on a full measure set, is compensated by the fact that the missing set of measure zero coincides, modulo at most countably many points, with the set where $\{\psi_i\}$ is equal to it's average $\overline{\psi}$ for which we have established that the pointwise derivative coincides with the distributional one. This last property, as explained in [Ta] page 76, permits to proceed to formal integration by parts and deduce for instance relations of the form : for any $w_0 \in D^2_{R_0}$ and almost every $r > 0$ such that $D^2_r(w_0) \subset D^2_{R_0}$ we have

$$\int_{D^2_r(w_0)\cap\tilde{U}} \sum_{i=1}^N |\overline{\partial}\psi_i|^2 = \frac{1}{4}\int_{D^2_r(w_0)\cap\tilde{U}} \sum_{i=1}^N |\nabla\psi_i|^2 - \frac{\sqrt{-1}}{4}\int_{\partial D^2_r(w_0)} \sum_{i=1}^N (\psi_i \cdot d\overline{\psi}_i)$$

(which is an elementary relation as long as one deals with a single valued $W^{1,2}$ function, but which has to be checked carefully in the present setting). In view of these integration by parts being true and the almost $\overline{\partial}-$ equation solved by $\{\psi_i\}$ one can transpose standard unique continuation arguments to the present setting of multivalued function to prove properties such as isolation of the zeros of $\{\psi_i - \overline{\psi}\}$. This is done in [Ta] page 76-82 and since the zero set of $\{\psi_i - \overline{\psi}\}$ coincides with $\text{Sing}^N_\star u \cup \text{Sing}^N_0 u$, we deduce the following result

Lemma III.7 *$Sing^N u \setminus Sing^{N-1} u$ is made of isolated points in M^4.*

In order to fully establish proposition III.1 it remains to prove that a point in $\text{Sing}^N u \setminus \text{Sing}^{N-1} u$ cannot be an accumulation point of $\text{Sing}^{N-1} u$ and also to prove the smoothness of the preimages of regular values p by u while crossing a singularity in $\text{Sing}^N u \setminus \text{Sing}^{N-1} u$.

End of the proof of proposition III.1.

1) $\text{Sing}^N u$ has no accumulation points. The proof of this fact follows closely the proof of lemma 6.18 of [Ta]. We explain briefly how Taubes arguments can be adapted to our setting. Let $x_0 \in \text{Sing}^N u \setminus \text{Sing}^{N-1} u$ and consider two multivalued functions $\{\psi_i^l\}_{i=1\cdots N}$, $l = 1, 2$, whose graphs give respectively $u^{-1}_{r,x_0}(p^1)$ and $u^{-1}_{r,x_0}(p^2)$ for two given regular values p^1 and p^2 of u. In fact, because of the induction hypothesis, the smoothness of $u^{-1}(p)$, for regular p, through singular points in $\text{Sing}^{N-1} u$ combined with the proposition II.4 gives the uniqueness of the tangent cone (which was only an "almost uniqueness" in proposition II.7). Consider the map $\xi : u^{-1}_{r,x_0}(p^1) \cup u^{-1}_{r,x_0}(p^2) \setminus \{0\} \to \mathbb{C}P^1$ introduced in the beginning of section III.2 that assigns to any x in $u^{-1}_{r,x_0}(p^1) \cup u^{-1}_{r,x_0}(p^2) \setminus \{0\}$ the X whose corresponding $J_{r,x_0}-$holomorphic foliation Θ^X is such that the Θ^X_w passing through x is tangent to $u^{-1}_{r,x_0}(p^1) \cup u^{-1}_{r,x_0}(p^2)$ at x when it exists. For any

$x \in \mathrm{Sing}^{N-1}u$ (there is an at most countable family of it, say $x_1, x_2 \cdots$) we choose a foliation (i.e. a system of coordinate Θ^X) so that X is a regular value for ξ (as before) but also, so that $X \neq \xi(x_1), \xi(x_2) \cdots$ in order to have a $J_{r,x_0}-$holomorphic foliation which is transverse to the tangent planes at the singularities in $\mathrm{Sing}^{N-1}u$. Recall that we denote $\tilde{U}$ the dense open set given by $D^2 \setminus \pi(\mathrm{Sing} u_{r,x_0}) \cup \{w_1^T, w_2^T \cdots\}$ where $y_i = (w_i^T, z_i^T)$ are the points in $\overline{u_{r,x_0}^{-1}(p^1)} \cup \overline{u_{r,x_0}^{-1}(p^2)} \setminus \mathrm{Sing} u$ where the D_w and $\overline{u_{r,x_0}^{-1}(p^1)} \cup \overline{u_{r,x_0}^{-1}(p^2)} \setminus \mathrm{Sing} u$ do not intersect each other transversally. Because of Lemma III.7 and the induction hypothesis, for r small enough, we know that $[\mathrm{Sing} u_{r,x_0} \cap B_1^4] \cup \{(w_i^T, z_i^T)\}_i$ is made of at most countably many points with possibly one possible accumulation point : $(0,0)$. We mix ψ^1 and ψ^2 to construct a single $2N-$valued function that we call ψ. On $\tilde{U}$ we introduce the unique valued function given by

$$f(w) := \prod_{i \neq k} (\psi_i(w) - \psi_k(w))$$

From properties such as the one given by lemma III.6, f extends continuously to the whole of $D_{R_0}^2$, moreover, because of Lemma III.1, it is in $W^{1,2}(D_{R_0}^2)$. From the study above we have $\pi(\mathrm{Sing} u_{r,x_0}) \subset f^{-1}(0)$. We shall prove now that $\pi(\mathrm{Sing}^{N-1} u_{r,x_0})$ is finite. Using the induction hypothesis $u_{r,x_0}^{-1}(p)$, in a neighborhood of a point $x_1 \in \mathrm{Sing}^{N-1} u_{r,x_0}$ is a smooth $J_{r,x_0}-$holomorphic curve for which x_1 can be a branch point. Since the tangents of $u_{r,x_0}^{-1}(p^1) \cup u_{r,x_0}^{-1}(p^2)$ are transverse to the planes W =constant (this is ensured by the appropriate choice of $J-$holomorphic foliation Θ^X we made above) we can use the explicit normal form of a $J_{r,x_0}-$holomorphic curve at a branch point given by the lemma 7.1 of [MW] to check that the index of f at such a zero is strictly positive. Therefore, in order to prove the claim that $\pi(\mathrm{Sing}^{N-1} u_{r,x_0})$ is finite, it suffices to show that for any zero w_0 of f and for almost every radius r, the degree of $f/|f|$ on $\partial D_r^2(w_0)$ is non-negative.

On a simply connected subset Δ of $\tilde{U}$ we can find a globally defined indexation for ψ so that each ψ_i, for $i = 1 \cdots 2N$, is smooth. Given a pair $i \neq k$, lemma III.2 gives

$$\frac{\partial(\psi_i - \psi_k)}{\partial \overline{z}} = \frac{1}{2} \left(\delta_r(\psi_i) + \delta_r(\psi_k)\right) \frac{\partial(\psi_i - \psi_k)}{\partial z}$$
$$+ \frac{1}{2} \left[\frac{\partial \psi_i}{\partial z} + \frac{\partial \psi_k}{\partial z}\right] \frac{\delta_r(\psi_i) - \delta_r(\psi_k)}{\psi_i - \psi_k} (\psi_i - \psi_k) + \frac{\nu_r(\psi_i) - \nu_r(\psi_k)}{\psi_i - \psi_k} (\psi_i - \psi_k)$$

Let $h^{ik}_r := \frac{1}{2}[\delta_r(\psi_i)+\delta_r(\psi_k)]$ and $k^{ik}_r := \left[\frac{\partial\psi_i}{\partial z}+\frac{\partial\psi_k}{\partial z}\right]\frac{\delta_r(\psi_i)-\delta_r(\psi_k)}{\psi_i-\psi_k}+2\frac{\nu_r(\psi_i)-\nu_r(\psi_k)}{\psi_i-\psi_k}$. Using these notations we get

$$\begin{aligned}\frac{\partial f}{\partial\overline{z}} &= \sum_{i<k}\frac{f}{(\psi_i-\psi_k)^2}\ h^{ik}_r\ \frac{\partial(\psi_i-\psi_k)^2}{\partial\overline{z}}+\sum_{i<k}k^{ik}_r\ f\\ &= \left[\sum_{i<k}h^{ik}_r\right]\ \frac{\partial f}{\partial z}+\left[\sum_{i<k}k^{ik}_r\right]\ f\end{aligned} \tag{III.24}$$

Observe that both $\sum_{i<k}h^{ik}_r$ and $\sum_{i<k}k^{ik}_r$ are symmetric in the indices and therefore extend as well defined functions on the whole $\tilde{U}$. Since the distributional derivative of f coincides with the pointwise derivative ($f\in W^{1,2}(D^2_{R_0})$) and since $\tilde{U}$ is an open dense subset of $D^2_{R_0}$, the PDE (III.24) solved by f extends on the whole $D^2_{R_0}$. Using the behavior of δ_r and ν_r given by lemma III.2 and the control of the Morrey-norm of $\nabla\psi_i$ given by lemma 6.15 of [Ta] we deduce that, in one hand, we have

$$\lim_{r\to 0}\|\sum_{i<k}h^{ik}_r\|_{L^\infty}=0 \tag{III.25}$$

in the other hand

$$\|\sum_{i<k}k^{ik}_r\|_{L^2(D^2_r)}\le C\ r^{\frac{N}{2}} \tag{III.26}$$

Let w_0 be a zero of f. We have to show that the degree $d^\rho_{w_0}$ of $f/|f|$ on $\partial D^2_\rho(w_0)$ for generic ρ, below a given value, is non-negative. Recall that the zeros of f are included in $\text{Sing}^{N-1}u\cup\{0\}\cup\{w^T_1,w^T_2\cdots\}$ which is made of at most countably many points which are isolated except possibly 0. Choose ρ so that $\partial D^2_\rho(w_0)$ avoids this set. Choose now ε being a regular value of $|f|$ (which is smooth on the part of the domain where $|f|>0$) less than the inf of $|f|$ on $\partial D^2_\rho(w_0)$. Call $\Delta_\varepsilon := \{x\in D^2_\rho(w_0)\ ;\ |f|\le\varepsilon\}$ and $\gamma_\varepsilon=\partial\Delta_\varepsilon$. Let $\varepsilon'<\varepsilon/2$ being another regular value of $|f|$. We have

$$\int_{\{x\in\Delta_\varepsilon\ ;\ |f|>\varepsilon'\}}\frac{i}{2}df\wedge d\overline{f}=\int_{\{x\in\Delta_\varepsilon\ ;\ |f|>\varepsilon'\}}|\partial f|^2-|\overline{\partial}f|^2 \tag{III.27}$$

The coarea formula gives

$$[\varepsilon^2-\varepsilon'^2]\,d_{w_0}=\int_{\varepsilon'}^{\varepsilon}ds\int_{|f|=s}\chi_{\Delta_\varepsilon}\ \overline{f}df=\int_{\{x\in\Delta_\varepsilon\ ;\ |f|>\varepsilon'\}}idf\wedge d\overline{f} \tag{III.28}$$

where $\chi_{\Delta_\varepsilon}$ is the characteristic function of the set Δ_ε. Combining (III.24)...(III.28) we get

$$[\varepsilon^2 - \varepsilon'^2]\, d_{w_0} \geq \int_{\{x\in\Delta_\varepsilon\ ;\ |f|>\varepsilon'\}} (1 - \|\sum_{i<k} h_r^{ik}\|_\infty)\ |\partial f|^2 - C\varepsilon^2\rho^N \quad , \quad \text{(III.29)}$$

where $\rho > 0$ is such that $\Delta_\varepsilon \subset D^2_\rho(w_0)$. For ρ small enough, this clearly implies that $d_{w_0} \geq 0$. This proves that $\text{Sing}^{N-1} u_{r,x_0}$ is finite in B^4_1. ■

Smoothness of $u^{-1}(p)$ in $B^4 \setminus \text{Sing}^{N+1} u$.

Given $x_0 \in \text{Sing}^N u \setminus \text{Sing}^{N-1} u$ and $r > 0$, it remains to prove that, for r small enough and almost every $p \in \mathbb{C}P^1$, $u^{-1}_{r,x_0}(p) \cap B^4_1$ is a smooth $J_{r,x_0}-$holomorphic curve in B^4_1. Here r is chosen small enough so that $\text{Sing}\, u_{r,x_0} \cap B^4_1 = \{0\}$. Choose p to be a regular value of u_{r,x_0} in $B^4_1 \setminus \{0\}$. $u^{-1}_{r,x_0}(p)$ is given by a $N-$valued graph $\{(w, \psi_i(w))\}_{i=1\cdots N}$ for suitable coordinates constructed above. Let S_0 be the regular complex surface given by $u^{-1}_{r,x_0}(p) \cap B^4_1 \setminus \{0\}$ equipped with the complex structure j given by J_{r,x_0}. This surface has a boundary $u^{-1}_{r,x_0}(p) \cap \partial B^4_1$ and annular ends diffeomorphic to $[0,\infty) \times S^1$ given by the image by $\{\psi_i\}_{i=1\cdots N}$ of the annuli $D^2_r \setminus \{0\}$ for r as small as we want. The restriction to $u^{-1}_{r,x_0}(p) \cap B^4_1 \setminus \{0\}$ of the metric $g_{r,x_0} := g(rx + x_0)$ (we are still using the normal coordinates about x_0 in which we obtained $\{\psi_i\}$) gives a metric on S_0, compatible with the complex structure j. In this metric, the distance from every point of the surface $u^{-1}_{r,x_0}(p) \cap B^4_1 \setminus \{0\}$ its collar ends is finite. From the Lipshitz property of the multivalued graph $\{(w, \psi_i(w))\}_{i=1\cdots N}$ established in Lemma III.6, the length of the set of points distant by ρ from a given collar end is comparable to ρ as ρ becomes small (linear growth). Therefore there exists a conformal diffeomorphism Λ sending (S_0, j) to the complement of finitely many points $\{a_1 \cdots a_K\}$ in a compact Riemann surface (Σ, j_Σ).

The conformal embedding that sends $(\Sigma, j_\Sigma) \setminus \{a_1 \cdots a_K\}$ into B^4_1 via Λ^{-1} and the inclusion embedding of S_0 into $u^{-1}_{r,x_0}(p) \cap B^4_1 \setminus \{0\}$ extends continuously to the whole of Σ into a weak conformal map into (B^4_1, g_{r,x_0}) having finite $W^{1,2}$ energy (since $\mathcal{H}^2(u^{-1}_{r,x_0}(p)) < +\infty$). Therefore it is a harmonic map from $\Sigma \setminus \{a_1 \cdots a_K\}$ into (B^4_1, g_{r,x_0}) having finite $W^{1,2}$ energy and therefore, from the removability singularity result of Sacks-Uhlenbeck, (theorem 3.6 of [SaU]) or from the regularity theory of harmonic map ([He]), it extends to a smooth conformal map on the whole of Σ. We have proved that $u^{-1}(p) \cap B_r(x_0)$ is a smooth $J-$holomorphic curve for r small enough. ■

IV Proof of theorem I.1 for $N = \mathbb{C}P^n$, $n > 1$.

Given $a := [1, \lambda_1, \cdots, \lambda_n]$ in $\mathbb{C}P^n$, we consider the holomorphic projection

$$\begin{array}{rcl} \pi_a \ : \ \mathbb{C}P^n & \longrightarrow & \mathbb{C}P^{n-1} \\ & & \\ [z_0, z_1 \cdots z_n] & \longrightarrow & [z_1 - \lambda_1 z_0, \cdots, z_n - \lambda_n z_0] \end{array}$$

which is singular at a and coincides with the identity on $\mathbb{C}P^{n-1} \subset \mathbb{C}P^n$ where $\mathbb{C}P^{n-1}$ in $\mathbb{C}P^n$ is identified with the point of the form $[0, z_1 \cdots z_n]$. Let U be the complement of $\mathbb{C}P^{n-1} \subset \mathbb{C}P^n$. We have the following lemma

Lemma IV.1 *Given u a $J-$holomorphic map in $W^{1,2}(M^4, \mathbb{C}P^n)$ which is locally approximable, then for almost every a in U, $\pi_a \circ u$ is a locally approximable $J-$holomorphic map in $W^{1,2}(M^4, \mathbb{C}P^{n-1})$*

Proof of lemma IV.1. Let u satisfying the assumption of the lemma. We have, for any $a \in U$,

$$|\nabla(\pi_a \circ u)|^2(x) \le C \frac{|\nabla u|^2}{|u(x) - a|^2}$$

Integrating this identity first in a over U and then in x over M^4 we get, using the fact that $dim\ \mathbb{C}P^n > 2$,

$$\int_{x \in M^4} \int_{a \in U} |\nabla(\pi_a \circ u)|^2(x) \le C \int_{x \in M^4} \int_{a \in U} \frac{|\nabla u|^2(x)}{|u(x) - a|^2} \le C \int_{x \in M^4} |\nabla u|^2 < +\infty \tag{IV.1}$$

Therefore, using Fubini Theorem, we deduce that for almost every a in U $\pi_a \circ u \in W^{1,2}(M^4, \mathbb{C}P^{n-1})$. Since u is a locally approximable $J-$holomorphic map, using theorem II.3, u is smooth outside a closed $\mathcal{H}^2-$measure 0 set $\mathrm{Sing}\, u$ and for almost all $a \in U$, $u^{-1}(a)$ is made of either at most countably many points (for $n = 2$) accumulating on $\mathrm{Sing}\, u$ or the empty set. In any cases, $\pi_a \circ u$ is smooth on a full measure open set and is $J-$holomorphic on it. Then for almost every a, $\pi_a \circ u$ solves (I.1) almost everywhere on M^4. Therefore for almost every a, $\pi_a \circ u$ is a $J-$holomorphic map in $W^{1,2}(M^4, \mathbb{C}P^{n-1})$. It remains to show that it is locally approximable for almost every a.

Because of (I.2), for any ball $B^4 \subset M^4$ there exists $u_k \in C^\infty(B^4; \mathbb{C}P^n)$ such that

$$\int_{B^4} |\nabla(u_k - u)|^2 \longrightarrow 0 \tag{IV.2}$$

Almost every $a \in U$ is a regular value for the whole sequence u_k (with $u_k^{-1}(a) \neq \emptyset$ for a.e. a only for $n = 2$). Therefore, for a. e. a, $\pi_a \circ u_k$ has at most finitely many singular points in B^4 (no singular point for $n > 2$) and since $\pi_a \circ u_k \in W^{1,2}$ for a.e. $a \in U$ (because of the argument above), it is not difficult to check that

$$\text{for a. e. } a \text{ and } \forall k \qquad d\left((\pi_a \circ u_k)^* \omega_{\mathbb{C}P^n}\right) = 0 \tag{IV.3}$$

(local obstructions to the strong approximability in $W^{1,2}(B^4, \mathbb{C}P^{n-1})$ are boundary of codimension 2 rectifiable currents - see [GMS]). We claim that for a.e. a we can extract a subsequence $u_{k'}$

$$\pi_a \circ u_{k'} \longrightarrow \pi_a \circ u \qquad \text{strongly in } W^{1,2} \tag{IV.4}$$

We have used the isometrical embedding of $\mathbb{C}P^n$ into some $\mathbb{R}^k$ to define $W^{1,2}(M^4; \mathbb{C}P^n)$, and we get

$$\begin{aligned} |\nabla(\pi_a \circ u_k - \pi_a \circ u)|^2 &\leq C \left| \sum_j \frac{\partial \pi_a}{\partial y_j}(u_k)\, \nabla(u_k^j - u^j) \right|^2 \\ &+ C \sum_j \left| \frac{\partial \pi_a}{\partial y_j}(u) - \frac{\partial \pi_a}{\partial y_j}(u_k) \right|^2 \ |\nabla u|^2 \\ &\leq \frac{C}{|u_k - a|^2} |\nabla(u_k - u)|^2 + C \frac{|u_k - u|^2}{|u_k - a|^2\ |u - a|^2}\ |\nabla u|^2 \end{aligned} \tag{IV.5}$$

Integrating the above inequality on $a \in U$ we get

$$\int_{a \in U} |\nabla(\pi_a \circ u_k - \pi_a \circ u)|^2 \leq C |\nabla(u_k - u)|^2 + C |u_k - u|^2\ \left|\log |u_k - u|\right|\ |\nabla u|^2 \tag{IV.6}$$

And we deduce that

$$\lim_{k \to +\infty} \int_{a \in U} \int_{B^4} |\nabla(\pi_a \circ u_k) - \nabla(\pi_a \circ u)|^2 \longrightarrow 0 \tag{IV.7}$$

Then we deduce that for almost every a in U there exists a subsequence $u_{k'}$, depending possibly on a, such that $(\pi_a \circ u_{k'}) \to (\pi_a \circ u)$ in $W^{1,2}$. Since the equation $d\left((\pi_a \circ u_{k'})^* \omega_{\mathbb{C}P^n}\right) = 0$ passes to the limit for strong convergences in $W^{1,2}$, we have established that

$$\text{for a.e. } a \in U \qquad d(u^* \omega_{\mathbb{C}P^n}) = 0 \quad \text{in } \mathcal{D}'(B^4)$$

We use an at most countable covering of M^4 by balls B^4 to extend the previous assertion to $\mathcal{D}'(M^4)$. We have then established that for a. e. $a \in U$ $\pi_a \circ u$ is a locally approximable $J-$holomorphic map in $W^{1,2}(M^4, \mathbb{C}P^{n-1})$. This proves lemma IV.1. ∎

Proof of theorem I.1 for $N = \mathbb{C}P^n$, $n > 1$.

We prove theorem I.1 by induction on n. We assume theorem I.1 holds for $n-1$. Then, by means of lemma IV.1 above, we deduce that for a. e. $a \in U$, $\pi_a \circ u$ is smooth except at isolated points in M^4.

Assume x_0 is a singularity of u which has disappeared while projecting u by some π_a on $\mathbb{C}P^{n-1}$. Since $\pi_a \circ u$ is now continuous at x_0, both the tangent holomorphic map $\phi : \mathbb{C}P^1 \to \mathbb{C}P^n$ at x_0 and the bubbling at x_0 along the $J_{x_0}-$holomorphic disks given by the proposition II.4, have to take values in the $\mathbb{C}P^1$ given by $\pi_a^{-1}((\pi_a \circ u)(x_0))$. Choosing a' in U which is not in $\pi_a^{-1}((\pi_a \circ u)(x_0))$ will therefore maintain x_0 as being a singularity for $\pi_{a'} \circ u$. Granting this remark we argue as follows : let $\text{Sing}_\pi u$ be the set of singularities of u that disappears while composing u with a π_a for one of the generic $a \in U$ for which lemma IV.1 holds (call $\tilde{U}$ this set). To any $x_0 \in \text{Sing}_\pi u$ we assign the $\mathbb{C}P^1 \subset \mathbb{C}P^n$ into which both ϕ and the bubbling takes value at x_0, call it $\mathbb{C}(x_0)$. As we explained above, for any $a \in \tilde{U} \setminus \mathbb{C}(x_0)$, $\pi_a \circ u$ is singular at x_0. Assume for contradiction that $\text{Sing}_\pi u \cap K$ is infinite for some compact subset K of M^4. Since any $\mathbb{C}(x_0)$ is 2-dimensional and since $\tilde{U}$ is a positive measure set in $\mathbb{C}P^n$, we may choose $a = a_\infty \in \tilde{U}$ that avoids infinitely many $\mathbb{C}(x_0)$ for x_0 in $\text{Sing}_\pi u \cap K$ and we would therefore have an infinite singular set for $\pi_{a_\infty} \circ u$ that would contradicts the induction hypothesis. Therefore $\text{Sing}\, u$ is finite in any compact subset of M^4 and theorem I.1 is proved. ∎

A Appendix

Lemma A.1 *Let J be a C^∞ almost complex structure on the unit ball B^4 of $\mathbb{R}^4$, then there exists a radius $1 > r > 0$ and a C^∞ 2-form Ω on $B_r(0)$ such that*

i)

$$d\Omega = 0$$

ii)

$$\forall X \in \mathbb{R}^4 \ X \neq 0 \qquad \Omega(X, JX) > 0$$

iii)

$$\forall X, Y \in \mathbb{R}^4 \quad \Omega(JX, JY) = \Omega(X, Y)$$

Proof of Lemma A.1 : Modulo a change of coordinate we may always assume that J at the origin coincide with the standard complex structure on $\mathbb{R}^4 \simeq \mathbb{C} \oplus \mathbb{C}$, $J(0) := \frac{\partial}{\partial x_2} \oplus dx_1 - \frac{\partial}{\partial x_1} \oplus dx_2 + \frac{\partial}{\partial x_4} \oplus dx_3 - \frac{\partial}{\partial x_3} \oplus dx_4$. The standard symplectic form $\Omega_0 := dx_1\, dx_2 + dx_3\, dx_4$ then verifies ii) in a neighborhood of the origin. Let $\hat{\Omega}$ defined by $\hat{\Omega}(X, Y) := \Omega_0(X, Y) + \Omega_0(JX, JY)$. It clearly verifies ii) and iii). We shall now multiply $\hat{\Omega}$ by a positive function in order to have the closedness condition to be solved. Assume $d\hat{\Omega}(0) = 0$ then consider $\tilde{\Omega} := e^{x_1}\hat{\Omega}$ at 0 we have $d\tilde{\Omega}(0) = dx_1 \wedge dx_3 \wedge dx_4$. Therefore we may assume that $d\hat{\Omega} \neq 0$ in a neighborhood of 0. From [Ol] we know that any rank 2 non-closed forms in 4 dimensions are locally equivalent, therefore there exists a system of coordinates (x, y, z, t) about 0 such that $\hat{\Omega} = e^x\, [dx \wedge dy + dz \wedge dt]$. Multiplying $\hat{\Omega}$ by e^{-x} the closedness condition is ensured and ii) and iii) still hold. ∎

We keep denoting J_0 the standard complex structure on $\mathbb{R}^4 \simeq \mathbb{C} \oplus \mathbb{C}$. We recall that $\mathbb{C}P^1$ parameterizes the space of complex lines in $\mathbb{C} \oplus \mathbb{C}$ passing through the origin. We will denote D^2_r the disc of $\mathbb{C}$ centered at the origin with radius r. We will first prove the following lemma.

Lemma A.2 *For any $\varepsilon > 0$ there exists $\alpha > 0$ such that for any J almost complex structure on $D^2_2 \times D^2_2$ verifying*

i) $J_{x_0} = J_0$.

ii) $\|J - J_0\|_{C^{l,\nu}} \leq \alpha$

($l \in \mathbb{N}$, $l > 2$ and $0 < \nu < 1$). Then, for any $X = [W, Z]$ in $\mathbb{C}P^1$ verifying

$$|W| \leq |Z| \tag{A.1}$$

there exists an embedded $J-$holomorphic disk D_X in $\mathbb{R}^4$ such that *i)* $D_X = \{(w,z) \quad ; \quad w = h_X(z)\}$ *and*

$$\left\| h_X(z) - \frac{W}{Z} z \right\|_{C^{l,\nu}} \leq \varepsilon$$

ii) $T_0 D_X = X$

Moreover : for any $x = (w,z) \in D_1^2 \times D_1^2 \setminus \{(0,0)\}$ *such that* $|w| \leq |z|/2$, *there exists a unique* X *in* $\mathbb{C}P^1$ *satisfying (A.1) such that* $x \in D_X$.

Proof of Lemma A.2. For $X = [W,Z]$ in $\mathbb{C}P^1$ satisfying (A.1), we denote

$$D_{0,X} = \left\{ (w,z) \quad ; \quad w = h^0_{0,X}(z) = \frac{W}{Z} z \right\}$$

We are first looking for a perturbation $\tilde{h}_X(z) = h^0_{0,X}(z) + \tau_{0,X}(z)$ of $h^0_{0,X}(z)$ such that $z \to \tilde{H}_X(z) = (\tilde{h}_X(z), z)$ realizes a $J-$holomorphic graph. We first look for a perturbation $\hat{H}_X(z) = (h^0_{0,X}(z) + \lambda(z); z + \mu(z))$ of $H^0_{0,X}(z) = (h^0_{0,X}(z), z)$ so that $\hat{H}_X$ realizes an holomorphic map into (B^4, J). We require $\hat{H}_X$ to solve

$$\frac{\partial \hat{H}_X}{\partial y} = J \frac{\partial \hat{H}_X}{\partial x} = i \frac{\partial \hat{H}_X}{\partial x} + (J - J_0) \frac{\partial \hat{H}_X}{\partial x}$$

which writes

$$\frac{\partial T_X}{\partial \overline{z}} = i(J - J_0) \frac{\partial H^0_{0,X}}{\partial x} + i(J - J_0) \frac{\partial T_X}{\partial x} \tag{A.2}$$

where $\hat{H}_X = H^0_{0,X} + T_X$. The following problem

$$\left\{ \begin{array}{l} \dfrac{\partial u}{\partial \overline{z}} = F \qquad \text{in } D_1^2 \\[2ex] u_{|_{\partial D_1^2}} \in \text{Span}\left\{ e^{-i\theta k} \quad ; \quad k \in \mathbb{Z}^*_+ \right\} \end{array} \right. \tag{A.3}$$

is a well posed elliptic problem from $C^{l,\nu}(D^2)$ into $C^{l+1,\nu}(D^2)$. Therefore, for $\|(J - J_0)\|_{C^{l,\nu}(D^2)}$ small enough, we may proceed to a standard fixed point argument in these spaces to get T_X solving (A.2). μ is small in $C^{l+1}-$norm therefore $\zeta \ : \ z \to z + \mu(z)$ is invertible. We take $\tilde{H}_X(z) := \hat{H}_X \circ \zeta^{-1}$.

The difficulty is that the $J-$holomorphic graph $\tilde{H}_X$ we just constructed above does not necessarily pass through the origin. In order to overcome this difficulty we extend the previous construction as follows.

For any $p = (w_p, z_p)$ in B^4 and for $X = [W, Z]$ in $\mathbb{C}P^1$ satisfying (A.1), we denote

$$D^0_{p,X} = \left\{(w, z) \quad ; \quad w = h^0_{p,X}(z) = \frac{W}{Z}(z - z_p) + w_p\right\}$$

We first look for a perturbation $\hat{H}_{p,X}(z) = (h_{p,X}(z) + \lambda_{p,X}(z); z + \mu_{p,X}(z))$ of $H^0_{p,X}(z) = (h^0_{p,X}(z), z)$ being a holomorphic into (B^4, J). We require $\hat{H}_{p,X}$ to solve

$$\frac{\partial \hat{H}_{p,X}}{\partial y} = J(\hat{H}_{p,X})\frac{\partial \hat{H}_{p,X}}{\partial x} = i\frac{\partial \hat{H}_{p,X}}{\partial x} + (J - J_0)(\hat{H}_{p,X})\frac{\partial \hat{H}_{p,X}}{\partial x}$$

which writes again

$$\frac{\partial T_{p,X}}{\partial \overline{z}} = i(J - J_0)(\hat{H}_{p,X})\frac{\partial H^0_{p,X}}{\partial x} + i(J - J_0)(\hat{H}_{p,X})\frac{\partial T_{p,X}}{\partial x} \tag{A.4}$$

where $\hat{H}_{p,X} = H^0_{p,X} + T_{p,X}$. This problem is solved like above where we obtained a solution of (A.2) satisfying $\|T_{p,X}\|_{C^{l+1,\nu}(D^2_1)} \leq C(\alpha)$ with $C(\alpha) \rightarrow 0$ as α tends to 0. Observe also that, once the existence of T is established, it is not difficult to obtain estimates of the form

$$\left\|\frac{\partial^k}{\partial X^s \partial p^t} T_{p,X}\right\|_{C^{l+1-k,\nu}(D^2_1)} \leq C(\alpha) \tag{A.5}$$

where $t+s = k$ and $\frac{\partial^k}{\partial X^s \partial p^t} T_{p,X}$ is a notation for any $k-th$derivatives along the parametric variables X and p in a given coordinate chart for the subpart $\mathbb{C}P^1$ given by (A.1). Denote finally $\tilde{H}_{p,Y} = \hat{H}_{p,Y} \circ \zeta^{-1}_{p,Y}$ where $\zeta_{p,Y}(z) = z + \mu_{p,Y}(z)$. Therefore $\tilde{H}_{p,Y}$ writes $\tilde{H}_{p,Y}(z) = (\tilde{h}_{p,Y}(z), z)$. We introduce the following map

$$\begin{aligned} \Psi \ : \ D^2 \times \mathbb{C}P^1 \quad &\longrightarrow \quad D^2 \times \mathbb{C}P^1 \\ (w, Y) \quad &\longrightarrow \quad \left(\tilde{h}_{p,Y}(0); \left[\frac{\partial \hat{H}_{p,Y}}{\partial z}(\zeta^{-1}_{p,X}(0))\right]\right) \end{aligned}$$

where $p = (w, 0)$ and $\left[\frac{\partial \hat{H}_{p,Y}}{\partial z}(\zeta_{p,X}^{-1}(0))\right]$ denotes the complex line of $\mathbb{C}P^1$ passing through the origin and the point $\frac{\partial \hat{H}_{p,Y}}{\partial z}(\zeta_{p,X}^{-1}(0))$ in $\mathbb{C} \oplus \mathbb{C}$. Observe that since $\hat{H}_{p,Y}$ and $\zeta_{p,Y}$ are respectively perturbations of $H^0_{p,Y}$ and the identity in D^2_1 and since $\frac{\partial H^0_{p,Y}}{\partial z}(0) = Y$, $\frac{\partial \hat{H}_{p,Y}}{\partial z}(\zeta_{p,X}^{-1}(0))$ is non zero . In fact $\Psi = Id + K$ is a perturbation of the map identity which is nothing but $(w, Y) \to \left(H^0_{p,Y}(z_p) = p, \left[\frac{\partial H^0_{p,Y}}{\partial z}(z_p)\right] = Y\right)$. From (A.4) we have that $\|\nabla K\|_{C^{l,\nu}} \leq C(\alpha)$. Therefore, for α small enough, as a perturbation of the identity, Ψ is a diffeomorphism from $D^2_1 \times U$ (where $U \subset \mathbb{C}P^1$ is made of the $X = [W, Z]$ solving (A.1)) and whose range can be taken as close as we want from $D^2_1 \times U$ by choosing α small enough. Then there exists smooth functions $w_X \in D^2$ and $Y_X \in \mathbb{C}P^1$ such that $\Psi(w_X, Y_X) = (0, X)$. We then finally choose

$$D_X := \{(w, z) \quad ; \quad w = h_X(z) := \tilde{h}_{p_X, Y_X}(z)\}$$

where $p_X := (w_X, 0)$. It is clear that D_X passes through the origin and that D_X is a $J-$holomorphic disk. Since $\hat{H}_{(w_X,0),Y_X}$ is a $J-$holomorphic map, since $\hat{H}_{(w_X,0),Y_X} \circ \zeta^{-1}_{(w_X,0),Y_X}(0) = \tilde{H}_{(w_X,0),Y_X}(w_X, 0) = (0, 0)$ and since J at the origin coincides with the canonical complex structure of $\mathbb{R}^4 \simeq \mathbb{C} \oplus \mathbb{C}$, $X = \frac{\partial \hat{H}_{p,Y}}{\partial z}(\zeta_{p,X}^{-1}(0))$ coincides with the tangent plane to D_X at the origin. Therefore D_X solves i) and ii) in the statement of the lemma.

It remains to prove the last statement of the lemma : the fact that D_X realizes a $J-$holomorphic foliation of the punctured cone

$$\mathcal{C} := \left\{(w, z) \in D^2_1 \times D^2_1 \setminus \{(0, 0)\} \quad ; \quad |w| < |z|/2\right\}$$

Given any point $q = (w_q, z_q)$ in the cone $\mathcal{C}$, we will show that there exists a unique $\tilde{h}_{\Psi^{-1}(0,X)}$ for X in U passing through q. By an abuse of notation we will write $\tilde{h}_X$ for $\tilde{h}_{\Psi^{-1}(0,X)}$. Introduce the map

$$\begin{aligned} \Xi_q \; ; \; U \quad &\longrightarrow \quad \mathbb{C}P^1 \\ X \quad &\longrightarrow \quad \left[\tilde{h}_X(z_q), z_q\right] \end{aligned}$$

We claim that for α small enough, independent on q, Ψ_q is a C^1 perturbation of the identity

$$\begin{array}{rcl} Id\ ;\ U & \longrightarrow & U \\ & & \\ X & \longrightarrow & \left[h^0_{0,X}(z_q), z_q\right] = X \end{array}$$

Only the words "independent of q" have to be checked. We use the chart $X = [W, Z] \to \frac{W}{Z}$ in U. We have to verify that for α small enough, independent on q, we have

$$\left\| \frac{\partial}{\partial X_i} \left[\frac{\tilde{h}_X(z_q)}{z_q} - \frac{h^0_{0,X}(z_q)}{z_q} \right] \right\|_\infty \leq \varepsilon \tag{A.6}$$

Denote $r_X = \tilde{h}_{\Psi^{-1}(0,X)} - h^0_{0,X}$, granting the estimates above, in particular (A.5), we have $\|\nabla_z \frac{\partial}{\partial X} r_X\|_\infty \leq C(\alpha)$ using this fact we deduce

$$\|\nabla(\Psi_q - Id)\|_\infty \leq \left\| \frac{\frac{\partial}{\partial X} r_X(z_q) - \frac{\partial}{\partial X} r_X(0)}{z_q} \right\|_\infty \leq \left\| \nabla_z \frac{\partial}{\partial X} r_X \right\|_\infty \leq C(\alpha)$$

Therefore Ψ_q is a diffeomorphism from U into an open set V that tends to U as $\alpha \to 0$. Let $X_q := \Psi_q^{-1}([w_q, z_q])$ it is the only X in V for which $\tilde{H}_X$ passes through q. This proves Lemma A.2

We also need the following variant of the previous lemma whose proof is contained in the proof of the previous lemma.

Lemma A.3 *For any $\varepsilon > 0$ there exists $\alpha > 0$ such that, for any $J-$holomorphic complex structure on $D^2_2 \times D^2_2$ verifying*

i) $J_{x_0} = J_0$

ii) $\|J - J_0\|_{C^{l,\nu}} \leq \alpha$

($l \in \mathbb{N}$, $l > 2$ and $0 < \nu < 1$) then for any $p = (w_p, z_p)$ and $q = (w_q, z_q)$ in $D^2_1 \times D^2_1$ ($p \neq q$) verifying

$$|w_p - w_q| \leq |z_p - z_q|$$

there exists an embedded $J-$holomorphic disk $D_{p,q}$ such that *i)* $D_{p,q} := \{(w,z) \quad ; \quad w = h_{p,q}(z)\}$ *and*

$$\left\| h_{p,q}(z) - (z - z_p)\frac{w_p - w_q}{z_p - z_q} - w_p \right\|_{C^{l,\nu}} \leq \varepsilon$$

ii) $p, q \in D_{p,q}$

Using similar techniques one proves the following lemma

Lemma A.4 *($J-$holomorphic foliations) For any $\varepsilon > 0$ there exists $\alpha > 0$ such that for any $J-$holomorphic structure on $D_2^2 \times D_2^2$ verifying*

$$\|J - J_0\|_{C^{l,\nu}} \leq \alpha$$

($l \in \mathbb{N}$, $l > 2$ and $0 < \nu < 1$) there exists a smooth map $\Theta : U \rightarrow D_1^2 \times D_1^2$ where U is a neighborhood of $(0,0)$ in B^4 such that Θ has the form $\Theta(w,z) = (\theta_w(z), z)$ and verifies the following conditions

i) Θ *realizes a diffeomorphism from U into $D_1^2 \times D_1^2$*

ii) $\{(\theta_w(z), z)\}$ *realizes a $J-$holomorphic disk for each w.*

iii) $\|\theta_w(\cdot) - w\|_{C^{l,\nu}} \leq \varepsilon$.

iv) $\theta_w(0) = w$

We also use the following "family version" of the previous result.

Lemma A.5 *(System of $J-$holomorphic foliations) There exists $1 > \mu_0 > 0$ such that for any $\varepsilon > 0$ there exists $\alpha > 0$ such that for any smooth map*

$$M \ : \ D_2^2 \times D_2^2 \longrightarrow Gl_4(\mathbb{R})$$

$$x \longrightarrow M_x$$

verifying

$$\|Id - M\|_{C^{l,\nu}} \leq \alpha$$

(Id denotes here the constant map into $Gl_4(\mathbb{R})$ that assigns the identity matrix) there exists a map

$$\Theta \ : \ U_1 \times D_2^2 \times D_2^2 \longrightarrow D_2^2 \times D_2^2$$

$$(X, w, z) \longrightarrow (\theta_w^X(z), z)$$

(where $U_t = \{X = [W, Z] \in \mathbb{C}P^1 \quad ; \quad |W| < t|Z|\}$) verifying

i) $\forall X \in U_{\mu_0}$, $\Theta^X := \Theta(X,\cdot,\cdot)$ *realizes a diffeomorphism from an open subset* V_X *of* $D_2^2 \times D_2^2$ *into* $D_1^2 \times D_1^2$.

ii) *The image of* $\Theta_w^X := \Theta(X,w,\cdot)$ *realizes a* $J = M\,J_0\,M^{-1}-$*holomorphic disk.*

iii) $(w,0) \in \Theta_w^X(D_1^2)$.

iv) $\left|\theta_w^X(z) - \frac{W}{Z}\,z - w\right\|_{C^{l,\nu}} \leq \varepsilon$ *(*$X = [W,Z]$*).*

v) *For any* $p \in D_1^2 \times D_1^2$ *and* $Y \in U_{\mu_0}$ *there exists a unique* $(X,w) \in U_1 \times D_2^2$ *such that* Θ_w^X *passes through* p *with tangent at* p *the* $J = M J_0 M^{-1}-$*holomorphic 2-plane* MYM^{-1}*. Furthermore the map*

$$\Xi \;:\; (p,Y) \;\longrightarrow (w,X)$$

is C^1.

Proof of Lemma A.5 : As in the proof of Lemma A.2 we perturb the J_0-holomorphic foliation $H^{0,X} \;:\; (w,z) \;\to\; (w+\frac{W}{Z}z, z)$ (where $X = [W,Z]$) by adding T^X obtained by the fixed point method described above so that the resulting maps $\hat{H}_w^X = (w + \frac{W}{Z}z + \lambda_w^X(z), z + \mu_w^X(z))$ is holomorphic into (B_1^4, J) and we introduce the map $\tilde{H}_w^X := \hat{H}_w^X \circ (\zeta_w^X)^{-1}(z) = (\tilde{h}_w^X(z), z)$. In order to ensure iii) we observe that $K^X \;:\; w \to \tilde{h}_w^X(0) = w + \lambda_w^X(\zeta^{-1}(0))$ is invertible and we finally choose $\Theta_w^X = \tilde{H}_{K^{-1}(w)}^X = (\theta_w^X(z), z)$. Consider now the map

$$\mathcal{L} \;:\; D_1^2 \times D_1^2 \times U_1 \times D_1^2 \times U_1 \longrightarrow D_2^2 \times U_2$$

$$(w_p, z_p, Y, w, X) \longrightarrow \left(\theta_w^X(z_p) - w_p, M_{\Theta_w^X(z_p)} \partial_z \hat{H}_{K^{-1}(w)}^X {\zeta_w^X}^{-1}(z_p) M_{\Theta_w^X(z_p)}^{-1} - Y\right)$$

where we are using the chart $X = [W,Z] : \to W/Z$ on U_2. This map is a perturbation of $\mathcal{L}_0$:$(w_p, z_p, Y, w, X) \to (X\,z_p + w - w_p, X - Y)$. We then apply an implicit function theorem to get that

$$\{(w_p, z_p, Y, w, X) \quad ; \quad \text{s.t. } \mathcal{L}(w_p, z_p, Y, w, X) = (0,0)\}$$

is given by the points $(w_p, z_p, Y, \Xi(w_p, z_p, Y))$, where Ξ is a C^1 function. Lemma A.5 is proved. ∎

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